

Concentration on Surfaces for a Singularly Perturbed Neumann Problem in Three-Dimensional Domains

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Abstract

We consider the following singularly perturbed elliptic problem

$$\varepsilon^2 \Delta \tilde{u} - \tilde{u} + \tilde{u}^p = 0, \quad \tilde{u} > 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{u}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in \mathbb{R}^3 with smooth boundary, ε is a small parameter, \mathbf{n} denotes the inward normal of $\partial\Omega$ and the exponent $p > 1$. Let Γ be a hypersurface intersecting $\partial\Omega$ in the right angle along its boundary $\partial\Gamma$ and satisfying a *non-degenerate condition*. We establish the existence of a solution u_ε concentrating along a surface $\tilde{\Gamma}$ close to Γ , exponentially small in ε at any positive distance from the surface $\tilde{\Gamma}$, provided ε is small and away from certain *critical numbers*. The concentrating surface $\tilde{\Gamma}$ will collapse to Γ as $\varepsilon \rightarrow 0$.

Key Words. Singularly perturbed problems, Concentrations, Modified Fermi Coordinates,.

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1 Introduction

We consider the following problem

$$\varepsilon^2 \Delta \tilde{u} - \tilde{u} + \tilde{u}^p = 0, \quad \tilde{u} > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary, ε is a small parameter, \mathbf{n} denotes the inward normal of $\partial\Omega$ and the exponent $p > 1$.

1.1 Background and assumptions

Problem (1.1) is known as a stationary equation of Keller-Segel system in chemotaxis [19]. It can also be viewed as a limiting stationary equation of Gierer-Meinhardt system in biological pattern formation [14]. Problem (1.1) has been studied extensively in recent years. See [30] for backgrounds and references.

In the pioneering papers, under the condition that p is subcritical, i.e., $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 < p < +\infty$ when $N = 2$, Lin, Ni and Takagi[19], Ni and Takagi[31]-[32], established the existence of a least-energy solution U_ε of (1.1) and showed that, for ε sufficiently small, U_ε has only one local maximum point $P_\varepsilon \in \partial\Omega$. Moreover, $H(P_\varepsilon) \rightarrow \max_{P \in \partial\Omega} H(P)$ as $\varepsilon \rightarrow 0$, where $H(\cdot)$ is the mean curvature of $\partial\Omega$. Such solutions are called boundary spike-layers.

Since then, many papers investigated further the solutions of (1.1) concentrating at one or multiple points of $\overline{\Omega}$. (These solutions are called **spike-layers**.) A general principle is that the location of interior spikes is determined by the distance function from the boundary. We refer the reader to the articles [3], [7], [9], [15], [38], and references therein. On the other hand, boundary spikes are related to the mean curvature of $\partial\Omega$. This aspect is discussed in the papers [4], [6], [8], [16], [18], [37], and references therein. A good review of the subject up to 2004 can be found in [30].

The question of constructing higher-dimensional concentration sets has been investigated only in recent years. It has been conjectured in [30] that for any $1 \leq k \leq N - 1$, problem (1.1) has a solution U_ε which concentrates on a k -dimensional subset of $\overline{\Omega}$. We mention some results that support such a conjecture.

In [25] and [26], Malchiodi and Montenegro proved that for $N \geq 2$, there exists a sequence of numbers $\varepsilon_\ell \rightarrow 0$ such that problem (1.1) has a solution U_{ε_ℓ} which concentrates at the whole boundary $\partial\Omega$ (or any component of $\partial\Omega$). In [23, 24], Malchiodi showed the concentration phenomena for (1.1) along a closed non-degenerate geodesic (i.e. codimension 2) of $\partial\Omega$ in three-dimensional smooth bounded domain Ω . Mahmoudi and Malchiodi in [22] proved a full general concentration of solutions along k -dimensional ($1 \leq k \leq N - 1$) non-degenerate minimal submanifolds of the boundary for $N \geq 3$ and $1 < p < \frac{N-k+2}{N-k-2}$.

In the above papers [22]-[26], the higher dimensional concentration set is *on* the boundary. A natural question is if there are solutions with high dimensional concentration set *inside* the domain. For two dimensional case, the authors [39, 40] consider problem (1.1) with solutions concentrating on curves near a nondegenerate line Γ' connecting the boundary of Ω at right angle. The meaning of nondegeneracy of the line Γ' can be defined similarly as the nondegeneracy of the hypersurface Γ in the sequel. The reader can also refer to the survey paper by J. Wei [36].

The main objective of the present paper is to extend the result in [39] and show the existence of concentration on interior hypersurfaces touching the boundary of $\Omega \subset \mathbb{R}^3$. We make the following assumptions and notations. The reader can also refer to the references [11, 34], as well as [13] for some basic geometric results.

(A1): Our candidate hypersurface $\Gamma \in \Omega$ is a *minimal* surface that satisfies the following assumptions: Γ is smooth and embedded in \mathbb{R}^3 , and intersects $\partial\Omega$ in the right angle along its boundary $\partial\Gamma = \bar{\Gamma} \cap \partial\Omega$, which is a simple close curve in \mathbb{R}^3 . Let Δ^Γ be the Laplace-Beltrami operator on Γ and k_1, k_2 the principal curvatures of Γ . As a submanifold of \mathbb{R}^3 , then we define the norm of the second fundamental form of Γ by

$$|A_\Gamma|^2 = k_1^2 + k_2^2,$$

and the mean curvature of Γ by $k = k_1 + k_2$ (with scalar 2). Note that $k = 0$ along Γ . We recall again that \mathbf{n} is the inward unit normal vector on $\partial\Omega$, and hence, it locally is also the unit normal vector of the curve $\partial\Gamma$ because of the perpendicularity between $\bar{\Gamma}$ and $\partial\Omega$. Note that we here assume that the orientation of $\partial\Gamma$ is induced from that of $\partial\Omega$. Since a curve on the surface $\partial\Omega$ is a geodesic if and only if its normal vector is parallel to the normal vector \mathbf{n} of $\partial\Omega$. Therefore, by defining

$$I(y) = -\left\langle \frac{\partial \mathbf{n}}{\partial \nu}, \nu \right\rangle, \quad y \in \partial\Gamma \subset \partial\Omega. \quad (1.2)$$

we know that $-I$ is the curvature of the geodesic on $\partial\Omega$ passing through $y \in \partial\Gamma$ in the normal direction $\nu(y)$ of Γ .

(A2): Let τ be the normal of the curve $\partial\Gamma$, which is also the restriction of \mathbf{n} on $\partial\Gamma$, I is denoted in (4.3). By defining an *eigenvalue problem*, which will play an important role in our considerations,

$$\Delta^\Gamma f + |A_\Gamma|^2 f = \lambda f \quad \text{in } \Gamma, \quad \partial f / \partial \tau + I f = 0 \quad \text{on } \partial\Gamma, \quad (1.3)$$

we say that Γ is *non-degenerate* if the eigenvalue problem (1.3) does not have zero eigenvalues. The reader can refer to [34] for some explanations and examples.

1.2 The profile function w

Before stating the main result, we introduce two functions w and Z . Let w be the unique (even) solution of

$$w'' - w + w^p = 0 \text{ and } w > 0 \text{ in } \mathbb{R}, \quad w'(0) = 0, \quad w(\pm\infty) = 0. \quad (1.4)$$

It is well known that the associated linearized eigenvalue problem,

$$h'' - h + pw^{p-1}h = \lambda h \quad \text{in } \mathbb{R}, \quad \int_{\mathbb{R}} h^2 = 1, \quad h \in H^1(\mathbb{R}) \quad (1.5)$$

possesses a unique positive eigenvalue λ_0 with a unique even and positive eigenfunction Z . This follows for instance from the analysis in [32]. In fact, we have

$$w(x) = C_p \left\{ \exp\left[\frac{(p-1)x}{2}\right] + \exp\left[\frac{-(p-1)x}{2}\right] \right\}^{\frac{-2}{p-1}}, \quad (1.6)$$

$$Z = \left[\int_{\mathbb{R}} w^{p+1} \right]^{-\frac{1}{2}} w^{p+1}, \quad \lambda_0 = \frac{1}{4}(p-1)(p+3). \quad (1.7)$$

It is easy to see that for $|x| \gg 1$

$$w(x) = C_p e^{-|x|} - \frac{2C_p}{p-1} e^{-p|x|} + O(e^{-(2p-1)|x|}), \quad (1.8)$$

$$w'(x) = -C_p e^{-|x|} + \frac{2pC_p}{p-1} e^{-p|x|} + O(e^{-(2p-1)|x|}), \quad (1.9)$$

$$Z(x) = \tilde{C}_p e^{-(p+1)|x|} - \frac{2(p+1)\tilde{C}_p}{p-1} e^{-2p|x|} + O(e^{-(3p-1)|x|}), \quad (1.10)$$

where

$$C_p = \left[\frac{(p+1)}{2} \right]^{\frac{1}{p-1}}, \quad \tilde{C}_p = \left[\frac{(p+1)}{2} \right]^{\frac{p+1}{p-1}} \left[\int_{\mathbb{R}} w^{p+1} dx \right]^{-\frac{1}{2}}.$$

1.3 Main Theorem

Our main theorem can be stated as the following:

Theorem 1.1. *Assume that the minimal hypersurface Γ satisfies the nondegeneracy condition (1.3). There exists a sequence of small parameters $\{\varepsilon_l\}_l$ such that problem (1.1) has a positive solution u_ε , still denoting ε_l by ε , concentrating along a surface $\tilde{\Gamma}$ near Γ . Near Γ , u_ε takes the form*

$$u_\varepsilon(\tilde{y}) \approx w \left(\frac{\text{dist}(\tilde{y}, \tilde{\Gamma})}{\varepsilon} \right), \quad (1.11)$$

where w denotes the unique positive solution of problem (1.4). Moreover, there exists a positive number c_0 such that u_ε satisfies globally,

$$u_\varepsilon(\tilde{y}) \leq \exp[-c_0 \varepsilon^{-1} \text{dist}(\tilde{y}, \tilde{\Gamma})],$$

and the surface $\tilde{\Gamma}$ will collapse to Γ as $\varepsilon \rightarrow 0$. □

To explain in a few words the difficulties we have encountered, let us assume for the moment that $\Omega = \mathbb{R} \times \Gamma$ is an infinite strip. In terms of the stretched coordinates $(s, z) = \varepsilon^{-1}(r, y)$ the equation would look near the surface approximately like

$$v_{ss} + \Delta^{\Gamma_\varepsilon} v - v + v^p = 0, \quad (s, z) \in \mathfrak{S} := \mathbb{R} \times \Gamma_\varepsilon.$$

The effects of curvature and of the boundary conditions are here neglected. The linearization of this problem around the profile $w(s)$ becomes

$$\phi_{ss} + \Delta^{\Gamma_\varepsilon} \phi - \phi + pw^{p-1}\phi = 0, \quad (s, z) \in \mathfrak{S},$$

Let $\rho_i, \omega_i, i = 1, 2, \dots$, denote the eigenvalues and eigenfunctions of $-\Delta^\Gamma$, then functions of the form

$$\phi_i^1 = w_s(s) \omega_i(\varepsilon z), \quad \phi_i^2 = Z(s) \omega_i(\varepsilon z),$$

are eigenfunctions associated to eigenvalues respectively $-\rho_i \varepsilon^2$ and $\lambda_0 - \rho_i \varepsilon^2$. Many of these numbers are *small* and thus “*near non-invertibility*” of the linear operator occurs. These two effects, combined in principle orthogonally because of the L^2 -orthogonality of Z and w_s , are actually coupled through the smaller order terms neglected.

In [1, 20, 33], related singularly perturbed problems, the translation effect ϕ_i^1 have been successfully treated through successive improvements of the approximation and fine spectral analysis of the actual linearized operator. In [25, 26] resonance phenomena similar to the “ ϕ_i^2 -effect” has been faced in the Neumann problem involving whole boundary concentration. In [22]-[24], [10], [35], [39]-[40], the concentration on a k -dimensional minimal submanifold, involving both ϕ_i^1 and ϕ_i^2 effects, has been treated via arbitrary high order approximations.

To prove Theorem 1.1, not only the same difficulties as that in the two dimensional case in [39]-[40] are encountered but also more obstruction appears. In other words, in the present paper we need to handle more delicate resonance phenomena due to the existence of ϕ_i^2 -effect in higher dimension, as well as the strong interaction between the concentration set and the boundary due to the homogeneous boundary condition in (1.1). Some words are in order to explain the methods to handle these difficulties.

As we have stated in the above, we first neglect the boundary condition. By the suitable rescaling of the variables $(s, z) = \varepsilon^{-1}(r, y)$ in a type of Fermi coordinates(cf. Lemma 2.2), we then try the inner approximate solution to the problem roughly in the form,

$$v(s, z) = w(s - f(\varepsilon z)) + \varepsilon e(\varepsilon z) Z(s - f(\varepsilon z)) + \tilde{\varphi}(s, z), \quad (1.12)$$

where Z is defined in (1.5), f and e are left as parameters, while $\tilde{\varphi}(s, z)$ is L^2 -orthogonal for each z both to $w_s(s - f(\varepsilon z))$ and $Z(s - f(\varepsilon z))$. Solving first in $\tilde{\varphi}$ a natural projected problem, the standard reduction procedure will implies that the resolution of the full problem will be reduced to a nonlinear, nonlocal second order system of differential equations in (f, e) . The nonlinear system is not solvable due to the ϕ_i^2 effect in the case of dimension $N = 3$. This shows that the approximation in (1.12) doesn't work well as N becomes large. Hence we must improve our

approximation by the recurrence method used in [21], [24]-[26], [35]. The principle is: the better the approximation, higher the chances of a correct inversion of the full problem to obtain a contraction mapping formulation of the nonlinear, nonlocal second order differential equations. To do that, we try the following form as our new approximation, (see [26])

$$v(s, z) = w(s - f(\varepsilon z)) + \varepsilon e(\varepsilon z)Z(s - f(\varepsilon z)) + \sum_{l=1}^3 \varepsilon^l \varphi_l(s, z).$$

The aim of adding the term $\sum_{l=1}^3 \varepsilon^l \varphi_l(s, z)$ is to cancel the error term till order $O(\varepsilon^4)$ such that our approximation is good enough. After very tedious but necessary computations we find that such φ_l 's must satisfy some differential equations(cf. (3.52), (3.53), (3.54)), whose solvability relies on the parameter f . So we need to improve our approximation further by subtle adjusting the location of the concentration set, namely we take the following form of inner approximate solution, (see [22], [35])

$$\mathcal{V} = w(x) + \varepsilon e(\varepsilon z)Z(x) + \sum_{l=1}^3 \varepsilon^l \varphi_l(x, \varepsilon z) \quad \text{with} \quad x = s - \sum_{l=0}^2 \varepsilon^l f_l(\varepsilon z). \quad (1.13)$$

To fulfill the objective we need to analyze the corresponding linear problem very carefully and conduct lots of computations. This was done in Section 3. Note that the authors in [35] also met the same situations for the concentration phenomena on high dimensional hypersurface of inhomogeneous Schrödinger equation on any N dimensional space.

Second, we know that the inner approximation in general does not satisfy the boundary condition in (1.1). Note that, for the two dimensional case of (1.1), by using the condition that the limit location of concentration set, say Γ' as before, connect the boundary $\partial\Omega$ orthogonally, the authors [39]-[40] used a type of local coordinates (previously used by M. Kowalczyk in [20]) in a neighborhood of Γ' to stretch the boundary $\partial\Omega$ and decompose the interaction between the concentration set and boundary $\partial\Omega$ in such a way that they can improve the approximation to satisfy the boundary condition up to order of $O(\varepsilon^2)$. Unfortunately, the coordinate system can not be extended to the present case. Here, in a neighborhood of the hypersurface Γ , we will set up another type of local coordinates in Lemma 2.1(used by K. Sakamoto in [34]), called **modified Fermi Coordinates** in this paper for the convenience of notion, such that we can add boundary corrections to the inner expansion in (1.13) and finally get a good local approximation. The reader can refer to 4 for more details on the construction of boundary correction layers.

In the rest paper we carry out the program outlined above, which leads to the complete proof of Theorem 1.1 by the infinite dimensional reduction method introduced in [10]. The organization of the paper is as follows. In Section 2, we introduce a local modified Fermi coordinates and then set up local formulation of problem (1.1) by suitable rescaling. Sections 3 and 4 are devoted to the

construction of a good approximate solution to solve the problem up to order $O(\varepsilon^4)$. Indeed, we first ignore the boundary condition in (1.1) and construct an inner approximation solution by the induction method in Section 3. In Section 4, we add the boundary correction layers to the inner expansion and get the final approximations involving unknown functions (f_2, e) . In Section 5, a gluing procedure from [10] reduces the nonlinear problem to a projected problem on the infinite strip \mathfrak{S} , while in Section 6 and 7, we show that the projected problem has a unique solution for the pair of (f_2, e) in a chosen region. The final step is to adjust the parameters f_2 and e which is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the pair (f_2, e) with boundary conditions. This is done in Sections 8 and 9.

Finally, we remark that our arguments can be easily extended to show the existence of interior concentration phenomena approaching minimal non-degenerate hypersurfaces and possessing interaction with boundary for problem (1.1) on bounded domain of general dimension, see Remarks 3.2 and 4.2. However, for the clearness of presentation, we here only consider the case with concentration on hypersurfaces in three dimension.

2 Local Formulation of Problems

The main objective of this subsection is to set up a local suitable coordinate system near the minimal hypersurface Γ and then express problem (1.1) in local form.

2.1 Geometric notions

Let $\gamma_0(\cdot) : \mathbb{D} \rightarrow \Gamma \subset \Omega$ be a smooth parameterization of the surface Γ , where Γ is the minimal surface in Section 1 and $\mathbb{D} := \{y \in \mathbb{R}^2 : |y| < 1\}$ is the unit disk. For further computational convenience, we will use an isothermal representation $\gamma_0 : \mathbb{D} \rightarrow \Gamma$. Namely, we use γ_0 that satisfies

$$\left| \frac{\partial \gamma_0}{\partial y^1} \right|^2 = \left| \frac{\partial \gamma_0}{\partial y^2} \right|^2 = \hbar^2(y), \quad \left\langle \frac{\partial \gamma_0}{\partial y^1}, \frac{\partial \gamma_0}{\partial y^2} \right\rangle = 0. \quad (2.1)$$

Therefore, the tangent vectors $\partial \gamma_0 / \partial y^1$ and $\partial \gamma_0 / \partial y^2$ have the same length $\hbar(y) > 0$ and are mutually orthogonal.

Recall the notations given in subsection 1.1. For our future setting-up of problem (1.1), here we provide a type of local coordinates modified from the standard Fermi coordinates, called **modified Fermi coordinates** for convenience of notions. This coordinate system was previously used by K. Sakamoto to describe the transition layer for Allen-Cahn equation in [34].

Lemma 2.1. *There exist constants $\delta > 0$, $r_0 > 0$, which depend only on Γ and $\partial\Omega$, and a smooth diffeomorphism*

$$\gamma(\cdot, \cdot) : (-r_0, r_0) \times \mathbb{D}_\delta \rightarrow \Omega_\delta^{r_0}, \quad (2.2)$$

where \mathbb{D}_δ and $\Omega_\delta^{r_0}$ are given in (2.6) and (2.7) such that

1. $\gamma(0, y) = \gamma_0(y)$ for $y \in \mathbb{D}$, $\gamma(r, y) \in \partial\Omega$ for $y \in \partial\mathbb{D}$, $|r| < r_0$;
2. $\frac{\partial\gamma}{\partial r}(0, y) = \nu(y)$ for $y \in \mathbb{D}$;
3. $\gamma(r, y)$ has the following expansion, as $r \rightarrow 0$

$$\gamma(r, y) = \gamma_0(y) + r\nu(y) + \frac{r^2}{2}q_1(y) + \frac{r^3}{6}q_2(y) + O(r^4), \quad y \in \mathbb{D}, \quad (2.3)$$

where $q_1(y)$ and $q_2(y)$ are vector functions orthogonal to $\nu(y)$;

4. If we write γ as $\gamma(r, \vartheta, \rho)$ in terms of the coordinate (r, ϑ, ρ) in (2.13), then the derivative along the inward unit normal vector \mathbf{n} of $\partial\Omega$ is expressed as

$$\frac{\partial}{\partial \mathbf{n}} = \frac{1}{\sqrt{g^{33}}} \left(g^{13} \frac{\partial}{\partial r} + g^{23} \frac{\partial}{\partial \vartheta} + g^{33} \frac{\partial}{\partial \rho} \right), \quad (2.4)$$

where at $(r, \vartheta, 0)$, we have the expression

$$\begin{aligned} g^{13}(r, \vartheta) &= -r \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-2} \left\langle q_1, \frac{\partial \gamma_0}{\partial \rho} \right\rangle + O(r^2), \\ g^{23}(r, \vartheta) &= 2r \left| \frac{\partial \gamma_0}{\partial \vartheta} \right|^{-2} \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-2} \left\langle \frac{\partial \gamma_0}{\partial \vartheta}, \frac{\partial \nu}{\partial \rho} \right\rangle + O(r^2), \\ g^{33}(r, \vartheta) &= \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-2} + 2r \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-4} \left\langle \frac{\partial \gamma_0}{\partial \rho}, \frac{\partial \nu}{\partial \rho} \right\rangle + O(r^2). \end{aligned} \quad (2.5)$$

Proof. The proof can be found in [34]. For the convenience of readers, we also provide the details here. We extend γ_0 smoothly to

$$\mathbb{D}_\delta := \{ y \in \mathbb{R}^2 : |y| < 1 + \delta \}, \quad (2.6)$$

for some fixed constant $\delta > 0$. The extension is still denoted by γ_0 and its image by Γ_δ . Let $\nu(y) \in \mathbb{R}^3$ be the unit normal of Γ_δ at $\gamma_0(y) \in \Gamma_\delta$. We now define a neighborhood $\Omega_\delta^{r_0}$ of Γ_δ by the Fermi coordinates

$$\Omega_\delta^{r_0} := \{ \tilde{y} \in \mathbb{R}^3 : \tilde{y} = \gamma_0(y) + r\nu(y), |r| < r_0, y \in \mathbb{D}_\delta \}. \quad (2.7)$$

Now we choose $r_0 > 0$ in (2.7) small enough so that

$$(\Omega_\delta^{r_0} \cap \Omega) \cap \left\{ \gamma_0(y) + r\nu(y) : |r| < r_0, |y| = 1 + \delta \right\} = \emptyset.$$

When we deal with the portion of $\partial\Omega$ in $\Omega_\delta^{r_0}$, we use the coordinate $(\vartheta, \rho) \in \partial\mathbb{D} \times [0, \delta) \subset \mathbb{D}$ to parameterize a small region nearby $\partial\Gamma$ in Γ , where the points $(\vartheta, 0)$ is sent to the boundary of surface $\partial\Gamma$ by γ_0 and there holds

$$\left\langle \frac{\partial\gamma_0}{\partial\vartheta}, \frac{\partial\gamma_0}{\partial\rho} \right\rangle = 0 \quad \text{for } \rho = 0. \quad (2.8)$$

We define the Fermi coordinates in (2.7) by $\bar{\gamma} : (-r_0, r_0) \times \mathbb{D}_\delta \mapsto \mathbb{R}^3$, i.e.

$$\bar{\gamma}(r, y) = \gamma_0(y) + r\nu(y).$$

Note that this coordinate system is not suitable for the boundary of Ω . Whence the main objective of this proof is to make a suitable modification of $\bar{\gamma}$.

Denote by S the preimage of $\Omega_\delta^{r_0} \cap \partial\Omega$:

$$S = \bar{\gamma}^{-1}(\Omega_\delta^{r_0} \cap \partial\Omega).$$

Since $\partial\Omega \perp \Gamma$ by **(A1)**, we have

$$S \perp_{\partial\mathbb{D}} (\{0\} \times \mathbb{D}). \quad (2.9)$$

We also denote by C the preimage of $\Omega_\delta^{r_0} \cap \Omega$:

$$C = \bar{\gamma}^{-1}(\Omega_\delta^{r_0} \cap \Omega),$$

and by $C(r)$ the r -slice of C :

$$C(r) = \{y \in \mathbb{D}_\delta : (r, y) \in C\} \quad \text{for } |r| < r_0.$$

For later use, we set

$$C_\delta = \bar{\gamma}^{-1}(\Omega_\delta^{r_0}), \quad C_\delta(r) = \{y \in \mathbb{D}_\delta : (r, y) \in C_\delta\} \quad \text{for } |r| < r_0,$$

Since $\partial\Omega$ and γ_0 are smooth, $C(r)$ is a smooth domain, diffeomorphic to $C(0) = \mathbb{D}$. Therefore, there exists a family of smooth diffeomorphisms

$$\tilde{Y}(r, \cdot) : \mathbb{D} \mapsto C(r)$$

parametrized smoothly by $r \in (-r_0, r_0)$. Thanks to (2.9), we can choose \tilde{Y} so that

$$\tilde{Y}(0, y) = y, \quad \frac{\partial\tilde{Y}}{\partial r}(0, y) = 0, \quad y \in \mathbb{D}.$$

Furthermore, we make a smooth extension

$$Y(r, \cdot) : \mathbb{D}_\delta \mapsto C_\delta(r)$$

such that

$$\begin{aligned} Y(0, y) &= y, & \frac{\partial Y}{\partial r}(0, y) &= 0, & y &\in \mathbb{D}_\delta, \\ Y(r, y) &= \tilde{Y}(r, y), & (r, y) &\in (-r_0, r_0) \times \mathbb{D}. \end{aligned} \quad (2.10)$$

Let us now define the desired diffeomorphism γ by, called **modified Fermi coordinates**

$$\gamma(r, y) = \bar{\gamma}(r, Y(r, y)) = \gamma_0(Y(r, y)) + r\nu(Y(r, y)) \quad (2.11)$$

for $(r, y) \in (-r_0, r_0) \times \mathbb{D}_\delta$. It is now straightforward to verify that γ in (2.11) satisfies Lemma 2.1(1). By elementary computations and (2.10), we find that

$$\begin{aligned} \frac{\partial \gamma}{\partial r}(0, y) &= \nu(y) = \nu(Y(0, y)), \\ q_1(y) &= \frac{\partial^2 \gamma}{\partial r^2}(0, y) = \sum_{j=1}^2 \frac{\partial \gamma_0}{\partial Y^j} \frac{\partial^2 Y^j}{\partial r^2}(0, y) \perp \nu(y), \\ q_2(y) &= \frac{\partial^3 \gamma}{\partial r^3}(0, y) = \sum_{j=1}^2 \left(\frac{\partial \gamma_0}{\partial Y^j} \frac{\partial^3 Y^j}{\partial r^3}(0, y) + 3 \frac{\partial \nu}{\partial Y^j} \frac{\partial^2 Y^j}{\partial r^2}(0, y) \right) \perp \nu(y), \end{aligned} \quad (2.12)$$

which will show the validity of the statements (2) and (3).

To prove (4) of Lemma 2.1, we use the coordinates (ϑ, ρ) introduced in the above. Recall that $(\vartheta, \rho = 0)$ parametrizes $\partial\mathbb{D}$ and ρ is chosen so that

$$\left\langle \frac{\partial \gamma_0}{\partial \vartheta}, \frac{\partial \gamma_0}{\partial \rho} \right\rangle = 0 \quad \text{at } \rho = 0.$$

For $y \in \mathbb{D}$ near $\partial\mathbb{D}$, we express $\gamma(r, y)$ by

$$\gamma(r, y) = \gamma(r, \vartheta, \rho). \quad (2.13)$$

We also denote by $\mathbf{n}(r, \vartheta)$ the unit inward normal vector of $\partial\Omega$ at $\gamma(r, \vartheta, 0)$. Note that at $\rho = 0$ (i.e. on $\partial\Omega \cap \Omega_\delta^{r_0}$), vectors

$$\frac{\partial \gamma}{\partial r}, \frac{\partial \gamma}{\partial \vartheta}, \frac{\partial \gamma}{\partial \rho} \in \mathbb{R}^3,$$

constitute a basis for \mathbb{R}^3 . Hence $\mathbf{n}(r, \vartheta)$ is expressed as

$$\mathbf{n} = a \frac{\partial \gamma}{\partial r} + b \frac{\partial \gamma}{\partial \vartheta} + c \frac{\partial \gamma}{\partial \rho} \quad \text{at } \rho = 0, \quad (2.14)$$

where $c > 0$. Since $\left\{ \frac{\partial \gamma}{\partial r}, \frac{\partial \gamma}{\partial \vartheta} \right\}$ spans the tangent space of $\partial\Omega$ at $\gamma(r, \vartheta, 0)$, we have

$$\left\langle \frac{\partial \gamma}{\partial r}, \mathbf{n} \right\rangle = 0, \quad \left\langle \frac{\partial \gamma}{\partial \vartheta}, \mathbf{n} \right\rangle = 0, \quad \langle \mathbf{n}, \mathbf{n} \rangle = 1 \quad \text{at } \rho = 0. \quad (2.15)$$

From (2.14) and (2.15), we easily obtain

$$a = \frac{\tilde{g}^{13}}{\sqrt{\tilde{g}^{33}}}, \quad b = \frac{\tilde{g}^{23}}{\sqrt{\tilde{g}^{33}}}, \quad c = \frac{\tilde{g}^{33}}{\sqrt{\tilde{g}^{33}}} = \sqrt{\tilde{g}^{33}},$$

where the formulas depend on the inverse of metric matrix at $\rho = 0$, i.e. $(\tilde{g}^{ij}) = (\tilde{g}_{ij})^{-1}$. More precisely, the metric coefficients g_{ij} 's are given by,

$$\begin{aligned}\tilde{g}_{11} &= \left\langle \frac{\partial \gamma}{\partial r}, \frac{\partial \gamma}{\partial r} \right\rangle, & \tilde{g}_{12} &= \tilde{g}_{21} = \left\langle \frac{\partial \gamma}{\partial r}, \frac{\partial \gamma}{\partial \vartheta} \right\rangle, \\ \tilde{g}_{22} &= \left\langle \frac{\partial \gamma}{\partial \vartheta}, \frac{\partial \gamma}{\partial \vartheta} \right\rangle, & \tilde{g}_{13} &= \tilde{g}_{31} = \left\langle \frac{\partial \gamma}{\partial r}, \frac{\partial \gamma}{\partial \rho} \right\rangle, \\ \tilde{g}_{33} &= \left\langle \frac{\partial \gamma}{\partial \rho}, \frac{\partial \gamma}{\partial \rho} \right\rangle, & \tilde{g}_{23} &= \tilde{g}_{32} = \left\langle \frac{\partial \gamma}{\partial \vartheta}, \frac{\partial \gamma}{\partial \rho} \right\rangle.\end{aligned}$$

Therefore, $\frac{\partial}{\partial \mathbf{n}}$ is given by

$$\frac{\partial}{\partial \mathbf{n}} = \frac{1}{\sqrt{\tilde{g}^{33}}} \left(\tilde{g}^{13} \frac{\partial}{\partial r} + \tilde{g}^{23} \frac{\partial}{\partial \vartheta} + \tilde{g}^{33} \frac{\partial}{\partial \rho} \right).$$

Let us now take Taylor expansions of $\tilde{g}^{jk}(r, \vartheta, 0)$ in r at $r = 0$. From the expansion of $\gamma(r, \vartheta, 0)$ in Lemma 2.1(3), we have, at $\rho = 0$

$$\begin{aligned}\frac{\partial \gamma}{\partial r} &= \nu(\vartheta, 0) + r q_1(\vartheta, 0) + \frac{r^2}{2} q_2(\vartheta, 0) + O(r^3), \\ \frac{\partial \gamma}{\partial \vartheta} &= \frac{\partial \gamma_0}{\partial \vartheta}(\vartheta, 0) + r \frac{\partial \nu}{\partial \vartheta}(\vartheta, 0) + \frac{r^2}{2} \frac{\partial q_1}{\partial \vartheta}(\vartheta, 0) + O(r^3), \\ \frac{\partial \gamma}{\partial \rho} &= \frac{\partial \gamma_0}{\partial \rho}(\vartheta, 0) + r \frac{\partial \nu}{\partial \rho}(\vartheta, 0) + \frac{r^2}{2} \frac{\partial q_1}{\partial \rho}(\vartheta, 0) + O(r^3).\end{aligned}\tag{2.16}$$

By using the orthogonalities $\frac{\partial \gamma_0}{\partial \vartheta} \perp \nu$, $\frac{\partial \nu}{\partial \vartheta} \perp \nu$, $q_1 \perp \nu$ and $\frac{\partial \gamma_0}{\partial \vartheta} \perp \frac{\partial \gamma_0}{\partial \rho}$, together with (2.16), we find that, at $\rho = 0$

$$\begin{aligned}g &= \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left| \frac{\partial \gamma_0}{\partial \vartheta} \right|^2 & 0 \\ 0 & 0 & \left| \frac{\partial \gamma_0}{\partial \rho} \right|^2 \end{pmatrix} \\ &\quad + r \begin{pmatrix} 0 & \langle q_1, \frac{\partial \gamma_0}{\partial \vartheta} \rangle & \langle q_1, \frac{\partial \gamma_0}{\partial \rho} \rangle \\ \langle q_1, \frac{\partial \gamma_0}{\partial \vartheta} \rangle & -2\tilde{L} & -2\tilde{M} \\ \langle q_1, \frac{\partial \gamma_0}{\partial \rho} \rangle & -2\tilde{M} & -2\tilde{N} \end{pmatrix} + O(r^2),\end{aligned}\tag{2.17}$$

where

$$\begin{aligned}-\tilde{L} &= \left\langle \frac{\partial \gamma_0}{\partial \vartheta}, \frac{\partial \nu}{\partial \vartheta} \right\rangle, & -\tilde{N} &= \left\langle \frac{\partial \gamma_0}{\partial \rho}, \frac{\partial \nu}{\partial \rho} \right\rangle, \\ -2\tilde{M} &= \left\langle \frac{\partial \gamma_0}{\partial \vartheta}, \frac{\partial \nu}{\partial \rho} \right\rangle + \left\langle \frac{\partial \gamma_0}{\partial \vartheta}, \frac{\partial \nu}{\partial \rho} \right\rangle.\end{aligned}$$

Therefore, we obtain

$$\begin{pmatrix} \tilde{g}^{11} & \tilde{g}^{12} & \tilde{g}^{13} \\ \tilde{g}^{21} & \tilde{g}^{22} & \tilde{g}^{23} \\ \tilde{g}^{31} & \tilde{g}^{32} & \tilde{g}^{33} \end{pmatrix} = \begin{pmatrix} 1 & \vdots & 0 & \vdots & 0 \\ 0 & \vdots & |\frac{\partial \gamma_0}{\partial \vartheta}|^{-2} & \vdots & 0 \\ 0 & \vdots & 0 & \vdots & |\frac{\partial \gamma_0}{\partial \rho}|^{-2} \end{pmatrix} \\ + r \begin{pmatrix} 0 & \vdots & -|\frac{\partial \gamma_0}{\partial \vartheta}|^{-2} \langle q_1, \frac{\partial \gamma_0}{\partial \vartheta} \rangle & \vdots & -|\frac{\partial \gamma_0}{\partial \rho}|^{-2} \langle q_1, \frac{\partial \gamma_0}{\partial \rho} \rangle \\ -|\frac{\partial \gamma_0}{\partial \vartheta}|^{-2} \langle q_1, \frac{\partial \gamma_0}{\partial \vartheta} \rangle & \vdots & 2|\frac{\partial \gamma_0}{\partial \vartheta}|^{-4} \tilde{L} & \vdots & 2|\frac{\partial \gamma_0}{\partial \vartheta}|^{-2} |\frac{\partial \gamma_0}{\partial \rho}|^{-2} \tilde{M} \\ -|\frac{\partial \gamma_0}{\partial \rho}|^{-2} \langle q_1, \frac{\partial \gamma_0}{\partial \rho} \rangle & \vdots & 2|\frac{\partial \gamma_0}{\partial \vartheta}|^{-2} |\frac{\partial \gamma_0}{\partial \rho}|^{-2} \tilde{M} & \vdots & 2|\frac{\partial \gamma_0}{\partial \rho}|^{-4} \tilde{N} \end{pmatrix} \\ + O(r^2).$$

This completes the proof of Lemma 2.1. \square

We can use the modified Fermi coordinate system in (2.11) to express the Laplacian operator Δ in (1.1) locally in terms of $(r, y) \in (-r_0, r_0) \times \mathbb{D}$. Since $\tilde{y} = \gamma(r, y)$, the standard metric in $\Omega_\delta^{r_0} \subset \mathbb{R}^3$ is pulled back to $g_{jk}(r, y)$:

$$g_{jk}(r, y) = \left\langle \frac{\partial \gamma}{\partial y^j}, \frac{\partial \gamma}{\partial y^k} \right\rangle, \quad j, k = 0, 1, 2, \quad (2.18)$$

where y^0 stands for r . Hence, the *Laplace-Beltrami operator* and *gradient operator* are defined in local coordinates by

$$\Delta \equiv \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j \right), \quad \nabla h \equiv g^{ij} \partial_i h \partial_j, \quad (2.19)$$

where h is any smooth function and the coefficients g^{ij} are the entries of the inverse matrix of $g = (g_{ij})$. There also hold similar expressions for the *Laplace-Beltrami operator* and *gradient operator* on Γ .

For later use, we also recall the Weyl's asymptotic formula, referring for example to [5], or to [17] and [29] for further details. Let $\rho_i, \omega_i, i = 1, 2, \dots$, denote the eigenvalues and eigenfunctions of $-\Delta^\Gamma$ (ordered to be non-decreasing in i and counted with the multiplicity), then we have that

$$\rho_i \rightarrow \frac{C i^{2/(N-1)}}{\text{Vol}(\Gamma)} \quad \text{as } i \rightarrow \infty, \quad (2.20)$$

where $\text{Vol}(\Gamma)$ is the volume of (Γ, \tilde{g}) and \tilde{g} is induced from g , C is a constant depending only on the dimension $N - 1$.

2.2 Local formulation of the scaled problem

If we set $u(\tilde{z}) = \tilde{u}(\varepsilon \tilde{z})$, then problem (1.1) is equivalent to the **scaled problem**

$$\Delta_{\tilde{z}} u - u + u^p = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \mathbf{n}_\varepsilon} = 0 \quad \text{on } \partial \Omega_\varepsilon. \quad (2.21)$$

After rescaling, we denote $\Omega_\varepsilon = \Omega/\varepsilon$ and also $\Gamma_\varepsilon = \Gamma/\varepsilon$.

To get local form of problem (2.21) and construct the approximation to a solution of (2.21), which concentrates near Γ_ε , after rescaling, we also introduce the scaled modified Fermi coordinates in the neighborhood of Γ_ε by

$$(s, z) = (s, z_1, z_2) \in \left(-\frac{r_0}{\varepsilon}, \frac{r_0}{\varepsilon}\right) \times \mathbb{D}_\varepsilon, \quad (2.22)$$

where $\mathbb{D}_\varepsilon := \{z \in \mathbb{R}^2 : |z| < 1/\varepsilon\}$ for some fixed constant $\varepsilon > 0$. More precisely, we denote

$$\gamma_\varepsilon(s, z) = \gamma(\varepsilon s, \varepsilon z)/\varepsilon. \quad (2.23)$$

However, for the portion of $\partial\Omega_\varepsilon$ in $\bar{\Omega}_\varepsilon$, we use the coordinate $(\theta, \eta) \in \partial\mathbb{D}_\varepsilon \times [0, \delta/\varepsilon) \subset \mathbb{D}_\varepsilon$ to parameterize a small region of $\partial\Gamma_\varepsilon$ in Γ_ε , where the points $(\theta, 0)$ is sent to the boundary of surface $\partial\Gamma_\varepsilon$ by γ_0 and there holds

$$\left\langle \frac{\partial\gamma_0}{\partial\theta}, \frac{\partial\gamma_0}{\partial\eta} \right\rangle = 0 \quad \text{for } \eta = 0. \quad (2.24)$$

As a direct consequence of Lemma 2.1, there holds

Lemma 2.2. *There exist a constant $r_0 > 0$, which depend only on Γ and $\partial\Omega$, such that*

1. $\gamma_\varepsilon(0, z) = \gamma_0(\varepsilon z)/\varepsilon$ for $z \in \mathbb{D}_\varepsilon$, $\gamma_\varepsilon(s, z) \in \partial\Omega_\varepsilon$ for $z \in \partial\mathbb{D}_\varepsilon$, $|s| < \frac{r_0}{\varepsilon}$;
2. $\frac{\partial\gamma_\varepsilon}{\partial s}(0, z) = \nu(\varepsilon z)$ for $z \in \mathbb{D}_\varepsilon$;
3. $\gamma_\varepsilon(s, z)$ has the following expansion, as $s \rightarrow 0$

$$\gamma_\varepsilon(s, z) = \gamma_0(\varepsilon z)/\varepsilon + s\nu(\varepsilon z) + \frac{\varepsilon s^2}{2}q_1(\varepsilon z) + \frac{\varepsilon^2 s^3}{6}q_2(\varepsilon z) + O(\varepsilon^3 s^4), \quad z \in \mathbb{D}_\varepsilon, \quad (2.25)$$

where $q_1(\varepsilon z)$ and $q_2(\varepsilon z)$ are vector functions defined in (2.12), which are orthogonal to $\nu(\varepsilon z)$.

4. If we write γ_ε as $\gamma_\varepsilon(s, \theta, \eta)$ in terms of the coordinate (s, θ, η) , then the derivative along the inward unit normal vector \mathbf{n}_ε of $\partial\Omega_\varepsilon$ is expressed as

$$\frac{\partial}{\partial \mathbf{n}_\varepsilon} = \frac{1}{\sqrt{g^{33}}} \left(g^{13} \frac{\partial}{\partial s} + g^{23} \frac{\partial}{\partial \theta} + g^{33} \frac{\partial}{\partial \eta} \right), \quad (2.26)$$

where at $(s, \theta, 0)$, we have the expression

$$\begin{aligned} g^{13}(s, \theta) &= -\varepsilon s \left| \frac{\partial\gamma_0}{\partial\rho} \right|^{-2} \left\langle q_1, \frac{\partial\gamma_0}{\partial\rho} \right\rangle + O(\varepsilon^2 s^2), \\ g^{23}(s, \theta) &= 2\varepsilon s \left| \frac{\partial\gamma_0}{\partial\vartheta} \right|^{-2} \left| \frac{\partial\gamma_0}{\partial\rho} \right|^{-2} \left\langle \frac{\partial\gamma_0}{\partial\vartheta}, \frac{\partial\nu}{\partial\rho} \right\rangle + O(\varepsilon^2 s^2), \\ g^{33}(s, \theta) &= \left| \frac{\partial\gamma_0}{\partial\rho} \right|^{-2} + 2\varepsilon s \left| \frac{\partial\gamma_0}{\partial\rho} \right|^{-4} \left\langle \frac{\partial\gamma_0}{\partial\rho}, \frac{\partial\nu}{\partial\rho} \right\rangle + O(\varepsilon^2 s^2). \end{aligned} \quad (2.27)$$

Proof. The reader can refer to Lemma 2.1 for the details of the proof. \square

Obviously, in the scaled modified Fermi coordinates, it is of importance to express Laplace-Beltrami operator in (2.21) in local form. In fact, we obtain the form by careful calculations

$$\Delta = \frac{\partial^2}{\partial s^2} + \Delta^{\Gamma_\varepsilon} + B_0(s, \varepsilon z) + B_1(s, \varepsilon z) + B_2(s, \varepsilon z), \quad (2.28)$$

where

$$\begin{aligned} B_0(s, \varepsilon z) &= \varepsilon k \frac{\partial}{\partial s} - \varepsilon^2 s (k_1^2(\varepsilon z) + k_2^2(\varepsilon z)) \frac{\partial}{\partial s} + \varepsilon^3 b(\varepsilon z) s^2 \frac{\partial}{\partial s}, \\ B_1(s, \varepsilon z) &= -\nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon} - 2s \nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon} \left(\frac{\partial}{\partial s} \right), \\ B_2(s, \varepsilon z) &= \varepsilon^2 \sum_{i=1}^2 a_i(\varepsilon s, \varepsilon z) \frac{\partial}{\partial z_i} + \varepsilon^2 \sum_{i=1}^2 b_i(\varepsilon s, \varepsilon z) \frac{\partial^2}{\partial z_i \partial s} + B_3(s, \varepsilon z). \end{aligned}$$

In the above, we have denoted

$$\nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon} = \varepsilon \hbar^{-2}(\varepsilon z) \sum_{j=1}^2 \left\langle q_1(0, \varepsilon z), \frac{\partial \gamma_0}{\partial y^j} \right\rangle \frac{\partial}{\partial z^j}.$$

The functions $b(\varepsilon z)$, $a_i(\varepsilon s, \varepsilon z)$, $b_i(\varepsilon s, \varepsilon z)$, $i = 1, 2$ satisfy:

$$\left| b(\varepsilon z), a_i(\varepsilon s, \varepsilon z), b_i(\varepsilon s, \varepsilon z) \right| \leq C(1 + |\varepsilon s|^4).$$

Note that $O(\varepsilon)$ -term in $B_0(s, \varepsilon z)$ is actually absent because Γ is minimal (cf. **(A1)**). Here we have denoted $\Delta^{\Gamma_\varepsilon}$ the Laplace-Beltrami operator on Γ_ε . The vector q_1 is defined in (2.25) and the differential operator $B_3(s, \varepsilon z)$ is of size $O(\varepsilon^4)$. Whence, by using of Lemma 2.2, we can get local form of problem (2.21). This will be done in more details in Sections 3 and 4.

3 Inner Approximate Solutions

In this section, we neglect the boundary condition in (2.21) and then find a local inner approximate solution to solve the first equation in (2.21) up to order of $O(\varepsilon^4)$. In fact, by the scaled modified Fermi coordinates in Lemma 2.2, we write down the local form of the first equation in problem (2.21) and then extend it to a differential equation with unknown functions defined on the infinite strip \mathfrak{S} in \mathbb{R}^3 with notations

$$\mathfrak{S} = \{ (x, z) : x \in \mathbb{R}, z \in \Gamma_\varepsilon \}, \quad \partial \mathfrak{S} = \{ (x, z) : x \in \mathbb{R}, z \in \partial \Gamma_\varepsilon \}. \quad (3.29)$$

3.1 Inner formulation of the scaled problem

As we have mentioned in subsection 2.2, in terms of the local coordinate system (s, z) (cf. (2.22)), in the neighborhood of Γ_ε , the differential equation in (2.21) is locally expressed as

$$u_{ss} + \Delta^{\Gamma_\varepsilon} u - u + u^p + B(u) = 0, \quad (3.30)$$

where the linear differential operator $B(u)$ is defined by

$$B(u) = B_0(u) + B_1(u) + B_2(u),$$

by the notations

$$\begin{aligned} B_0(u) &= -\varepsilon^2 s(k_1^2(\varepsilon z) + k_2^2(\varepsilon z)) \frac{\partial u}{\partial s} + \varepsilon^3 b(\varepsilon z) s^2 \frac{\partial u}{\partial s}, \\ B_1(u) &= -\nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon}(u) - 2s \nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon}\left(\frac{\partial u}{\partial s}\right), \\ B_2(u) &= \varepsilon^2 \sum_{i=1}^2 a_i(\varepsilon s, \varepsilon z) \frac{\partial u}{\partial z_i} + \varepsilon^2 \sum_{i=1}^2 b_i(\varepsilon s, \varepsilon z) \frac{\partial^2 u}{\partial z_i \partial s} + B_3(u). \end{aligned}$$

We assume that, in the (s, z) coordinates, the location of concentration of the solution is characterized by the surface

$$\tilde{\Gamma}_\varepsilon : s = \sum_{i=0}^2 \varepsilon^i f_i(\varepsilon z). \quad (3.31)$$

Remark 3.1. *The smooth functions f_0, f_1 are to be determined in Sections 3 and 4. More precisely, with the help of the validity of the nondegeneracy of Γ in subsection 1.1, we will choose f_0 by solving the equation (3.59) with boundary condition (4.11), and also the equation (3.63) with boundary condition (4.17) for f_1 . In fact, the nondegeneracy of Γ implies that f_0 is identically zero. While the unknown parameter f_2 is to be chosen by a type of reduction procedure, which is equivalent to solving a system of differential equations in Section 9 (cf. (9.1)-(9.4)).* \square

In the sequel, we always assume that f_2 satisfies the uniform constraint

$$\|f_2\|_a = \|f_2\|_{L^\infty(\Gamma)} + \|\nabla^\Gamma f_2\|_{L^\infty(\Gamma)} + \|\Delta^\Gamma f_2\|_{L^q(\Gamma)} \leq \varepsilon^{\frac{1}{2}}. \quad (3.32)$$

We consider a further changing of variables and define a new function $v(x, z)$ as follows

$$u(s, z) = v(x, z) \quad \text{with} \quad x = s - \sum_{i=0}^2 \varepsilon^i f_i(\varepsilon z), \quad z = z. \quad (3.33)$$

We now want to express the problem in the new coordinates. Whence we need the following formulas

$$\begin{aligned} u_s &= v_x, & u_{ss} &= v_{xx}, \\ \Delta^{\Gamma_\varepsilon}(u) &= \varepsilon^2 \left(\sum_{i=0}^2 \varepsilon^i \nabla^\Gamma f_i \right)^2 v_{xx} - \varepsilon^2 \left(\sum_{i=0}^2 \varepsilon^i \Delta^\Gamma f_i \right) v_x - \varepsilon \left(\sum_{i=0}^2 \varepsilon^i \nabla^\Gamma f_i \right) \nabla^{\Gamma_\varepsilon} v_x + \Delta^{\Gamma_\varepsilon} v, \\ \nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon}(u) &= -\varepsilon \left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{q_1}^\Gamma f_i \right) v_x + \varepsilon \nabla_{q_1}^{\Gamma_\varepsilon} v, & \nabla_{q_{1\varepsilon}}^{\Gamma_\varepsilon}(u_s) &= -\varepsilon \left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{q_1}^\Gamma f_i \right) v_{xx} + \varepsilon \nabla_{q_1}^{\Gamma_\varepsilon}(v_x), \\ \frac{\partial u}{\partial z_i} &= -\left(\sum_{j=0}^2 \varepsilon^{j+1} \frac{\partial f_j}{\partial y_i} \right) v_x + \frac{\partial v}{\partial z_i}, & \frac{\partial^2 u}{\partial z_i \partial s} &= -\left(\sum_{j=0}^2 \varepsilon^{j+1} \frac{\partial f_j}{\partial y_i} \right) v_{xx} + \frac{\partial^2 v}{\partial z_i \partial x}. \end{aligned}$$

Locally, this gives that u solves (3.30) if and only if the function v defined in (3.33) solves the following problem

$$S(v) \equiv v_{xx} + \Delta^{\Gamma_\varepsilon} v - v + v^p + B(v) = 0. \quad (3.34)$$

Note that we will consider problem (3.34) on the whole \mathfrak{S} (cf. (3.29)). In the above we have denoted the linear operator

$$B(v) = B_4(v) + B_5(v) + B_6(v). \quad (3.35)$$

The linear operator $B_4(v)$, $B_5(v)$ and $B_6(v)$ can be expressed explicitly by

$$\begin{aligned} B_4(v) &= \varepsilon^2 \left(\sum_{i=0}^2 \varepsilon^i \nabla^\Gamma f_i \right)^2 v_{xx} - \varepsilon^2 \left(\sum_{i=0}^2 \varepsilon^i \Delta^\Gamma f_i \right) v_x - \varepsilon \left(\sum_{i=0}^2 \varepsilon^i \nabla^\Gamma f_i \right) \nabla^{\Gamma_\varepsilon} v_x \\ &\quad - \varepsilon^2 \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) (k_1^2 + k_2^2) v_x + \varepsilon^3 b \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right)^2 v_x, \\ B_5(v) &= 2\varepsilon \left(x + \sum_{i=0}^2 \varepsilon^{i+1} f_i \right) \left[\left(\sum_{i=0}^2 \varepsilon^i \nabla_{q_1}^\Gamma f_i \right) v_{xx} - \nabla_{q_1}^{\Gamma_\varepsilon} (v_x) \right] + \varepsilon \left[\left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{q_1}^\Gamma f_i \right) v_x - \nabla_{q_1}^{\Gamma_\varepsilon} v \right], \\ B_6(v) &= \varepsilon^2 \sum_{i=1}^2 a_i \left[- \left(\sum_{j=0}^2 \varepsilon^{j+1} \frac{\partial f_j}{\partial y_i} \right) v_x + \frac{\partial v}{\partial z_i} \right] + \varepsilon^2 \sum_{i=1}^2 b_i \left[- \left(\sum_{j=0}^2 \varepsilon^{j+1} \frac{\partial f_j}{\partial y_i} \right) v_{xx} + \frac{\partial^2 v}{\partial z_i \partial x} \right] + B_3(v). \end{aligned}$$

3.2 Inner approximate solutions

In the subsection, we want to use coordinates (x, z) defined in (3.33) to construct a suitable approximation to a solution expressed in the form,

$$\mathcal{V} = w(x) + \varepsilon e(\varepsilon z) Z(x) + \sum_{i=1}^3 \varepsilon^i \varphi_i(x, \varepsilon z), \quad (3.36)$$

where w and Z are two functions given by (1.4) and (1.5). In the above expression, we have denoted φ_i , $i = 1, 2, 3$, smooth bounded functions to be determined in the sequel. As we have mentioned, the unknown parameters f_2 (cf. (3.31)) and e will be chosen in the last section by solving a system of differential equations (cf. (9.1)-(9.4)). In all what follows, we shall assume the validity of the following uniform constraints on the parameter e

$$\|e\|_b = \|e\|_{L^\infty(\Gamma)} + \varepsilon \|\nabla^\Gamma e\|_{L^q(\Gamma)} + \varepsilon^2 \|\Delta^\Gamma e\|_{L^q(\Gamma)} \leq \varepsilon^{\frac{1}{2}}. \quad (3.37)$$

For simplicity of notations, define

$$F = \left\{ (f_2, e) \mid \text{the functions } f_2 \text{ and } e \text{ satisfy (3.32) and (3.37) respectively} \right\}. \quad (3.38)$$

Now the key point is to choose suitable correction terms $\varphi_1, \varphi_2, \varphi_3$, and then prove that the approximate solution \mathcal{V} solve problem (3.34) up to order $O(\varepsilon^4)$.

Formally, we have

$$(w + \Theta)^p = w^p \left[1 + p \frac{\Theta}{w} + \frac{p(p-1)}{2} \left(\frac{\Theta}{w} \right)^2 + \dots + C_{k,p} \left(\frac{\Theta}{w} \right)^k + O\left(\left| \frac{\Theta}{w} \right|^{k+1} \right) \right]. \quad (3.39)$$

Setting $\Theta = \varepsilon(eZ + \varphi_1) + \sum_{i=2}^3 \varepsilon^i \varphi_i$ and separating the powers of ε , we get

$$\left(w + \varepsilon eZ + \sum_{i=1}^3 \varepsilon^i \varphi_i \right)^p = w^p \sum_{l=0}^4 \varepsilon^l \sum_{j_1, \dots, j_3, \sum ij_i=l} C_{l,j_1, \dots, j_3} \frac{(eZ + \varphi_1)^{j_1} \varphi_2^{j_2} \varphi_3^{j_3}}{w^{j_1 + \dots + j_3}} + w^p O\left(\left| \frac{\Theta}{w} \right|^5 \right).$$

Whence, using elementary calculation, we collect the powers of ε up to order 4 in the last formula, and then get the estimate

$$\begin{aligned} \mathcal{H}_0 &= \left| \left(w + \varepsilon eZ + \sum_{i=1}^3 \varepsilon^i \varphi_i \right)^p - w^p \sum_{l=0}^4 \varepsilon^l \sum_{j_1, \dots, j_3, \sum ij_i=l} C_{l,j_1, \dots, j_3} \frac{(eZ + \varphi_1)^{j_1} \varphi_2^{j_2} \varphi_3^{j_3}}{w^{j_1 + \dots + j_3}} \right| \\ &\leq C_{4,p} w^p \left[\varepsilon^5 \left(1 + \frac{|\Theta|}{w} \right)^4 + \left| \frac{\Theta}{w} \right|^5 \right]. \end{aligned}$$

More precisely, using (3.39), we make a decomposition

$$\begin{aligned} &w^p \sum_{l=0}^4 \varepsilon^l \sum_{j_1, \dots, j_3, \sum ij_i=l} C_{l,j_1, \dots, j_3} \frac{(eZ + \varphi_1)^{j_1} \varphi_2^{j_2} \varphi_3^{j_3}}{w^{j_1 + \dots + j_3}} \\ &= w^p + pw^{p-1} \varepsilon eZ + \sum_{i=1}^3 \varepsilon^i pw^{p-1} \varphi_i + \frac{1}{2} \varepsilon^2 p(p-1) w^{p-2} (eZ + \varphi_1)^2 \\ &\quad + p(p-1) w^{p-2} (eZ + \varphi_1) \sum_{i=3}^4 \varepsilon^i \varphi_{i-1} + \mathcal{H}_3 \\ &\equiv \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3, \end{aligned} \quad (3.40)$$

where we have denoted

$$\mathcal{H}_1 = w^p + pw^{p-1} \varepsilon eZ + \sum_{i=1}^3 \varepsilon^i pw^{p-1} \varphi_i, \quad (3.41)$$

$$\mathcal{H}_2 = \frac{1}{2} \varepsilon^2 p(p-1) w^{p-2} (eZ + \varphi_1)^2 + p(p-1) w^{p-2} (eZ + \varphi_1) \sum_{i=3}^4 \varepsilon^i \varphi_{i-1}. \quad (3.42)$$

In the above, we have also denoted that

$$\mathcal{H}_3 = \sum_{l=3}^4 \varepsilon^l \mathfrak{D}_l, \quad (3.43)$$

where for every $l = 3, 4$, the component \mathfrak{D}_l is independent of the terms $\varphi_{l-1}, \dots, \varphi_3$.

Putting \mathcal{V} into (3.34) and expanding formally, we derive that

$$\begin{aligned} &B(w) + \varepsilon(\varepsilon^2 \Delta^\Gamma e + \lambda_0 e)Z + \sum_{i=1}^3 \varepsilon^i [\varphi_{i,xx} - \varphi_i + pw^{p-1} \varphi_i] \\ &\quad + B(\varepsilon eZ) + \sum_{i=1}^3 \varepsilon^i \Delta^{\Gamma_\varepsilon} \varphi_i + \sum_{i=1}^3 \varepsilon^i B(\varphi_i) + \mathcal{H}_2 + \mathcal{H}_3 = 0. \end{aligned} \quad (3.44)$$

3.3 Inner errors

In this subsection, we accept that φ_i 's as known functions for the moment. Then we compute all error terms in (3.44) and make decomposition of all components into suitable forms according to the order of ε .

First of all, we calculate the term

$$B(w) = B_4(w) + B_5(w) + B_6(w). \quad (3.45)$$

Direct calculation gives that

$$B_4(w) + B_5(w) = \sum_{i=1}^4 \varepsilon^i \check{S}_i + \sum_{i=1}^4 \varepsilon^i \hat{S}_i + \sum_{i=5}^6 \varepsilon^i S_i,$$

with expressions defined by

$$\begin{aligned} \check{S}_1 &= \hat{S}_1 = 0, \\ \check{S}_2 &= -\Delta^\Gamma f_0 w_x - (k_1^2 + k_2^2) f_0 w_x, \\ \hat{S}_2 &= (\nabla^\Gamma f_0)^2 w_{xx} - (k_1^2 + k_2^2) x w_x + 2f_0 \nabla_{q_1}^\Gamma f_0 w_{xx}, \\ \check{S}_3 &= -\Delta^\Gamma f_1 w_x - (k_1^2 + k_2^2) f_1 w_x + b(f_0^2 + x^2) w_x, \\ \hat{S}_3 &= 2f_0 \nabla_{q_1}^\Gamma f_1 w_{xx} + 2f_1 \nabla_{q_1}^\Gamma f_0 w_{xx} + 2b f_0 x w_x, \\ \check{S}_4 &= -\Delta^\Gamma f_2 w_x - (k_1^2 + k_2^2) f_2 w_x, \\ \hat{S}_4 &= (\nabla^\Gamma f_1)^2 w_{xx} + 2f_0 \nabla_{q_1}^\Gamma f_2 w_{xx} + 2f_1 \nabla_{q_1}^\Gamma f_1 w_{xx} + 2f_2 \nabla_{q_1}^\Gamma f_0 w_{xx} + 2b f_1 x w_x. \end{aligned}$$

Note that for any $i = 1, \dots, 4$, \check{S}_i is an odd function in the variable x , while \hat{S}_i is an even function in the variable x . On the other hand, for any $i = 5, 6$, the high order term S_i is combination of powers of the parameters of f_0, f_1, f_2 and their derivatives with smooth bounded coefficients.

At the meantime, the linear operator $B_6(w)$ comes from $B_6(v)$ and can be expressed explicitly by

$$B_6(w) = \sum_{i=3}^4 \varepsilon^i \check{\mathbf{B}}_i(f_0, \dots, f_{i-3}) + \sum_{i=3}^4 \varepsilon^i \hat{\mathbf{B}}_i(f_0, \dots, f_{i-3}) + \varepsilon^5 \mathbf{B}_5(f_0, f_1, f_2),$$

where the terms $\mathbf{B}_5(f_0, f_1, f_2)$ and $\check{\mathbf{B}}_i(f_0, \dots, f_{i-3}), \hat{\mathbf{B}}_i(f_0, \dots, f_{i-3}), i = 3, 4$ are combination of powers of the parameters f_0, f_1, f_2 and their derivatives with smooth bounded coefficients. Moreover, $\check{\mathbf{B}}_i(f_0, \dots, f_{i-3}), i = 3, 4$ are odd functions in the variable x , while $\hat{\mathbf{B}}_i(f_0, \dots, f_{i-3}), i = 3, 4$ are even functions in the variable x . As a conclusion, we get

$$B(w) = \sum_{i=1}^4 \varepsilon^i \check{S}_i + \sum_{i=1}^4 \varepsilon^i \hat{S}_i + \sum_{i=5}^6 \varepsilon^i S_i + \sum_{i=3}^4 \varepsilon^i \check{\mathbf{B}}_i + \sum_{i=3}^4 \varepsilon^i \hat{\mathbf{B}}_i + \varepsilon^5 \mathbf{B}_5. \quad (3.46)$$

Second, we compute the error

$$\varepsilon(\varepsilon^2 \Delta^\Gamma eZ + \lambda_0 eZ) + B(\varepsilon eZ), \quad (3.47)$$

with $B(\varepsilon eZ) = B_4(\varepsilon eZ) + B_5(\varepsilon eZ) + B_6(\varepsilon eZ)$. There also holds

$$B_4(\varepsilon eZ) + B_5(\varepsilon eZ) + B_6(\varepsilon eZ) = \sum_{i=1}^3 \varepsilon^i \check{T}_i + \sum_{i=1}^3 \varepsilon^i \hat{T}_i + \sum_{i=4}^6 \varepsilon^i T_i.$$

In the above, we have denoted the following forms

$$\begin{aligned} \check{T}_1 &= \hat{T}_1 = 0, & \check{T}_2 &= \hat{T}_2 = 0, \\ \check{T}_3 &= -\Delta^\Gamma f_0 eZ_x - f_0(k_1^2 + k_2^2)eZ_x - \nabla^\Gamma f_0 \nabla^\Gamma eZ_x - 2f_0 \nabla_{q_1}^\Gamma eZ_x, \\ \hat{T}_3 &= (\nabla^\Gamma f_0)^2 eZ_{xx} - (k_1^2 + k_2^2)exZ_x - \nabla_{q_1}^\Gamma eZ + 2f_0 \nabla_{q_1}^\Gamma f_0 eZ_{xx} - 2\nabla_{q_1}^\Gamma exZ_x, \\ T_i &= \check{b}_i(f_0, f_1, f_2, e), \quad i = 4, 5, 6. \end{aligned}$$

In the above, for any $i = 4, 5, 6$, the term $\check{b}_i(f_0, f_1, f_2, e)$ is combination of powers of the parameter f_0, f_1, f_2, e and their derivatives with smooth bounded coefficients. Moreover, for any $i = 1, 2, 3$, \check{T}_i is an odd function in the variable x , while \hat{T}_i is an even function in the variable x .

In summary, we have that

$$\begin{aligned} S(\mathcal{V}) &= \sum_{i=1}^4 \varepsilon^i \check{S}_i + \sum_{i=1}^4 \varepsilon^i \hat{S}_i + \sum_{i=5}^6 \varepsilon^i S_i + \sum_{i=3}^4 \varepsilon^i \check{\mathbf{B}}_i + \sum_{i=3}^4 \varepsilon^i \hat{\mathbf{B}}_i \\ &\quad + \varepsilon^5 \mathbf{B}_5 + \sum_{i=1}^3 \varepsilon^i \check{T}_i + \sum_{i=1}^3 \varepsilon^i \hat{T}_i + \sum_{i=4}^6 \varepsilon^i T_i \\ &\quad + \varepsilon^3 \Delta^\Gamma eZ + \varepsilon \lambda_0 eZ + \sum_{i=1}^3 \varepsilon^i [\varphi_{i,xx} - \varphi_i + pw^{p-1} \varphi_i] \\ &\quad + \sum_{i=1}^3 \varepsilon^i \Delta^{\Gamma_\varepsilon} \varphi_i + \sum_{i=1}^3 \varepsilon^i B(\varphi_i) + \mathcal{H}_2 + \mathcal{H}_3. \end{aligned} \quad (3.48)$$

Now, we shall write the error terms involving correction terms $\varphi_1, \dots, \varphi_3$ in a suitable form. In the next subsection, for any given $i = 1, \dots, 3$, we will choose φ_i as the form

$$a_{i1}(\varepsilon z)b_{i1}(x) + a_{i2}(\varepsilon z)b_{i2}(x),$$

for some generic smooth functions a_{i1}, a_{i2}, b_{i1} (odd) and b_{i2} (even). Moreover, the terms a_{i1} and a_{i2} do not depend on the unknown parameters f_0, f_1 and f_2 . The reader can refer to (3.60) and (3.64).

Whence, we make a decomposition as

$$\sum_{i=1}^3 \varepsilon^i \Delta^{\Gamma_\varepsilon} \varphi_i = \sum_{j=3}^4 \varepsilon^j \check{\mathcal{N}}_j + \sum_{j=3}^4 \varepsilon^j \hat{\mathcal{N}}_j + \varepsilon^5 \mathcal{N}_5. \quad (3.49)$$

where

$$\begin{aligned}\check{\mathcal{N}}_j &= \check{\mathcal{N}}_j(\varphi_1, \dots, \varphi_{j-2}, f_0, \dots, f_{j-3}), \\ \hat{\mathcal{N}}_j &= \hat{\mathcal{N}}_j(\varphi_1, \dots, \varphi_{j-2}, f_0, \dots, f_{j-3}), \\ \mathcal{N}_5 &= \mathcal{N}_5(\varphi_1, \dots, \varphi_3, f_0, \dots, f_{k-2}).\end{aligned}$$

Moreover, $\check{\mathcal{N}}_j$ is an odd function in the variable x , while $\hat{\mathcal{N}}_j$ is an even functions in the variable x .

From the definition of the operator B in (3.34), we also write

$$\sum_{i=1}^3 \varepsilon^i B(\varphi_i) = \sum_{j=3}^4 \varepsilon^j \check{G}_j + \sum_{j=3}^4 \varepsilon^j \hat{G}_j + \sum_{j=5}^{10} \varepsilon^j G_j,$$

where, for any $j = 3, 4$, the components \check{G}_j and \hat{G}_j do not depend on the correction terms $\varphi_{j-1}, \dots, \varphi_3$ and the unknown parameters f_{j-2}, \dots, f_2 . In other words, we have

$$\begin{aligned}\check{G}_j &= \check{G}_j(\varphi_1, \dots, \varphi_{j-2}, f_0, \dots, f_{j-3}), \\ \hat{G}_j &= \hat{G}_j(\varphi_1, \dots, \varphi_{j-2}, f_0, \dots, f_{j-3}), \\ G_j &= G_j(\varphi_1, \dots, \varphi_3, f_0, \dots, f_2).\end{aligned}$$

Moreover, \check{G}_j is an odd function in the variable x , while \hat{G}_j is an even function in the variable x .

For later use, using (3.42) and (3.43), we decompose \mathcal{H}_3 into even parts and odd parts, and then write $\mathcal{H}_2 + \mathcal{H}_3$ as

$$\begin{aligned}\mathcal{H}_2 + \mathcal{H}_3 &= \frac{1}{2} \varepsilon^2 p(p-1) w^{p-2} (eZ + \varphi_1)^2 \\ &\quad + p(p-1) w^{p-2} (eZ + \varphi_1) \sum_{i=3}^4 \varepsilon^i \varphi_{i-1} + \sum_{j=3}^4 \varepsilon^j (\check{\mathfrak{D}}_j + \hat{\mathfrak{D}}_j),\end{aligned}\tag{3.50}$$

For $j = 3, 4$, the components $\check{\mathfrak{D}}_j$ and $\hat{\mathfrak{D}}_j$ are independent of the terms $\varphi_{j-1}, \dots, \varphi_3$, i.e.

$$\check{\mathfrak{D}}_j = \check{\mathfrak{D}}_j(\varphi_1, \dots, \varphi_{j-2}), \quad \hat{\mathfrak{D}}_j = \hat{\mathfrak{D}}_j(\varphi_1, \dots, \varphi_{j-2}),\tag{3.51}$$

Moreover, $\check{\mathfrak{D}}_j$ is an odd function in the variable x , while $\hat{\mathfrak{D}}_j$ is an even function in the variable x .

3.4 Determinations of the inner correction terms φ_i 's

In this subsection, by a recurrence procedure, we will choose suitable parameters f_0, f_1 so that we can really find the correction terms φ_2, φ_3 and then improve the approximation. In fact, it will be shown that φ_1 is identically zero.

This can be done in the following way. It is worth mentioning that the term

$$\varepsilon^3 \Delta^\Gamma eZ + \varepsilon \lambda_0 eZ$$

lies in the approximate kernel of the linearized problem of problem (3.34) at \mathcal{V} . We ignore this term for the moment and then cancel other components of the error in (3.48) with order of ε lower than 4 by choosing suitable correction terms $\varphi_1, \dots, \varphi_3$. Whence, for given $y \in \Gamma$, we then consider the problems

$$\begin{aligned} \varphi_{1,xx} - \varphi_1 + pw^{p-1}\varphi_1 &= 0 \quad \text{in } \mathbb{R}, \\ \varphi_1(\pm\infty) &= 0, \quad \int_{\mathbb{R}} \varphi_1 w_x \, dx = 0; \end{aligned} \tag{3.52}$$

$$\begin{aligned} \varphi_{2,xx} - \varphi_2 + pw^{p-1}\varphi_2 &= -\check{S}_2 - \hat{S}_2 - \frac{1}{2}p(p-1)w^{p-2}(eZ + \varphi_1)^2 \quad \text{in } \mathbb{R}, \\ \varphi_2(\pm\infty) &= 0, \quad \int_{\mathbb{R}} \varphi_2 w_x \, dx = 0; \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} \varphi_{3,xx} - \varphi_3 + pw^{p-1}\varphi_3 &= -\check{S}_3 - \hat{S}_3 - p(p-1)w^{p-2}(eZ + \varphi_1)\varphi_2 - \mathcal{C}_3 \quad \text{in } \mathbb{R}, \\ \varphi_3(\pm\infty) &= 0, \quad \int_{\mathbb{R}} \varphi_3 w_x \, dx = 0. \end{aligned} \tag{3.54}$$

In the above, we have denoted $\mathcal{C}_3 = \check{\mathcal{C}}_3 + \hat{\mathcal{C}}_3$, with the odd part and even part given by

$$\check{\mathcal{C}}_3 = \check{\mathbf{B}}_3 + \check{T}_3 + \check{\mathcal{N}}_3 + \check{G}_3 + \check{\mathfrak{D}}_3, \quad \hat{\mathcal{C}}_3 = \hat{\mathbf{B}}_3 + \hat{T}_3 + \hat{\mathcal{N}}_3 + \hat{G}_3 + \hat{\mathfrak{D}}_3. \tag{3.55}$$

Using (3.52), it is easy to show $\varphi_1 = 0$ and we finish the first step. To proceed the second step and cancel the error terms of order $O(\varepsilon^2)$ for the improvement of the approximation, we should choose the correction term φ_2 by solving problem (3.53). For this purpose, first, we collect all terms of order $O(\varepsilon^2)$ in $S(\mathcal{V})$, which has the form $\varepsilon^2 \mathcal{A}_2$ with

$$\mathcal{A}_2 = \check{S}_2 + \hat{S}_2 + \frac{1}{2}p(p-1)w^{p-2}(eZ)^2.$$

We denote the odd part and even part respectively by $\varepsilon^2 \check{\mathcal{A}}_2$ and $\varepsilon^2 \hat{\mathcal{A}}_2$ with

$$\begin{aligned} \check{\mathcal{A}}_2 &= -\Delta^\Gamma f_0 w_x - (k_1^2 + k_2^2) f_0 w_x, \\ \hat{\mathcal{A}}_2 &= (\nabla^\Gamma f_0)^2 w_{xx} - (k_1^2 + k_2^2) x w_x + 2f_0 \nabla_p^\Gamma f_0 w_{xx} + \frac{1}{2}p(p-1)w^{p-2}(eZ)^2. \end{aligned}$$

Then, we consider the problem

$$\begin{aligned} -\varphi_{2,xx} + \varphi_2 - pw^{p-1}\varphi_2 &= \check{\mathcal{A}}_2 + \hat{\mathcal{A}}_2 \quad \text{in } \mathbb{R}, \\ \varphi_2(\pm\infty) &= 0, \quad \int_{\mathbb{R}} \varphi_2 w_x \, dx = 0, \end{aligned} \tag{3.56}$$

as it is well known, which is uniquely solvable provided that

$$\int_{\mathbb{R}} (\check{\mathcal{A}}_2 + \hat{\mathcal{A}}_2) w_x \, dx = 0. \tag{3.57}$$

In fact, using the fact that w is an even function in the variable x , we have

$$\int_{\mathbb{R}} \hat{\mathcal{A}}_2 w_x \, dx = 0.$$

On the other hand, we get

$$\begin{aligned} \int_{\mathbb{R}} \check{\mathcal{A}}_2 w_x \, dx &= -\Delta^\Gamma f_0 \int_{\mathbb{R}} w_x^2 \, dx - (k_1^2 + k_2^2) f_0 \int_{\mathbb{R}} w_x^2 \, dx \\ &= -[\Delta^\Gamma f_0 + (k_1^2 + k_2^2) f_0] \int_{\mathbb{R}} w_x^2 \, dx. \end{aligned} \quad (3.58)$$

While (3.58) implies that

$$\int_{\mathbb{R}} \check{\mathcal{A}}_2 w_x \, dx = 0,$$

is equivalent to the following differential equation

$$\Delta^\Gamma f_0 + (k_1^2 + k_2^2) f_0 = 0 \quad \text{on } \Gamma. \quad (3.59)$$

In fact, we find that f_0 is identically zero by combining the equation (3.59) with the boundary condition (4.11) with the help of the non-degeneracy of Γ in (1.3). Whence, the solution to (3.53) can be expressed as

$$\varphi_2(x, z) = \psi_{21}(x, \varepsilon z) + \psi_{22}(x, \varepsilon z), \quad (3.60)$$

where $\psi_{21}(x, \varepsilon z)$ is in fact identically zero and $\psi_{22}(x, \varepsilon z)$ is an even function in the variable x . The components in $\psi_{21}(x, \varepsilon z)$ and $\psi_{22}(x, \varepsilon z)$ are independent of the parameters f_1 and f_2 .

In the same way, in order to cancel the error terms of order $O(\varepsilon^3)$ and improve the approximation by solving problem (3.54), we collect all terms of order $O(\varepsilon^3)$ in $S(\mathcal{V})$, which has the form $\varepsilon^3 \mathcal{A}_3$ with

$$\mathcal{A}_3 = \check{S}_3 + \hat{S}_3 + \frac{1}{2} p(p-1) w^{p-2} (eZ)^2 + p(p-1) w^{p-2} (eZ + \varphi_1) \varphi_2 + \mathcal{C}_3.$$

We denote the odd part and even part respectively by $\varepsilon^3 \check{\mathcal{A}}_3$ and $\varepsilon^3 \hat{\mathcal{A}}_3$. As the arguments in solving (3.56), we need an orthogonality condition like (3.57). Hence, we compute the projection of $\check{\mathcal{A}}_3$ and $\hat{\mathcal{A}}_3$ onto the kernel of the operator $\partial^2/\partial x^2 - 1 + p w^{p-1}$, which is spanned by w_x . In fact, we obtain

$$\check{\mathcal{A}}_3 = -\Delta^\Gamma f_1 w_x - (k_1^2 + k_2^2) f_1 w_x + \check{\mathcal{C}}_3 + g(f_0).$$

Here $g(f_0)$ is a smooth bounded function independent of the unknown parameters f_1 and f_2 .

Since the term $\hat{\mathcal{A}}_3$ is even in the variable x and w_x is odd in the variable x , there holds

$$\int_{\mathbb{R}} \hat{\mathcal{A}}_3 w_x \, dx = 0.$$

On the other hand, using the same arguments as in (3.58), we get

$$\begin{aligned} \int_{\mathbb{R}} \check{\mathcal{A}}_3 w_x \, dx &= -\Delta^\Gamma f_1 \int_{\mathbb{R}} w_x^2 \, dx - (k_1^2 + k_2^2) f_1 \int_{\mathbb{R}} w_x^2 \, dx + \int_{\mathbb{R}} \check{\mathcal{C}}_3 w_x \, dx + \int_{\mathbb{R}} g w_x \, dx \\ &= -[\Delta^\Gamma f_1 + (k_1^2 + k_2^2) f_1] \int_{\mathbb{R}} w_x^2 \, dx + d_1(f_0, e, \varphi_1, \varphi_2). \end{aligned} \quad (3.61)$$

Here d_1 is a smooth bounded function independent of the unknown parameters f_1, f_2 and the correction terms φ_3 and is Lipschitz continuous with respect to its parameters. By setting

$$b_1 = \int_{\mathbb{R}} w_x^2 \, dx \quad (3.62)$$

we derive from (3.61) that

$$\int_{\mathbb{R}} \check{\mathcal{A}}_3 w_x \, dx = 0,$$

is equivalent to the following differential equation

$$\Delta^\Gamma f_1 + (k_1^2 + k_2^2) f_1 = \frac{d_1}{b_1} \quad \text{on } \Gamma. \quad (3.63)$$

Now, we also choose f_1 by combining the equation (3.63) with the boundary condition (4.17) with the help of the non-degeneracy condition (1.3). The solution to (3.54) can be expressed as

$$\varphi_3(x, z) = \psi_{31}(x, \varepsilon z) + \psi_{32}(x, \varepsilon z), \quad (3.64)$$

where $\psi_{31}(x, \varepsilon z)$ is an odd function in the variable x and $\psi_{32}(x, \varepsilon z)$ is an even function in the variable x . The component in $\psi_{31}(x, \varepsilon z)$ and $\psi_{32}(x, \varepsilon z)$ are independent of the parameter f_2 .

Remark 3.2. *The recurrence procedure we described above is the same as the arguments in [35]. Whence, we can improve the local approximate solution to solve the first equation in (2.21) up to $O(\varepsilon^m)$ for any positive integer m .*

4 Boundary Correction Layers and Further Improvement of Approximations

For any given $(f_2, e) \in F$, we have the inner expansion \mathcal{V} in (3.36). This approximation in general does not satisfy the boundary condition in (2.21). In order to improve the approximation, we need to write problem (2.21) in local coordinates in the neighborhood of $\partial\Gamma \in \mathbb{R}^3$, and then add boundary correction layers to the inner expansion (3.36). In fact, by recalling the scaled modified Fermi coordinates with variables θ, η, s in Lemma 2.2 and also the translated variable x in (3.33), we will extend the local form of problem (2.21) to the infinite strip $\mathbf{\Lambda}$ in \mathbb{R}^3 with notations

$$\begin{aligned} \mathbf{\Lambda} &= \{ (x, \theta, \eta) : x \in \mathbb{R}, (\theta, \eta) \in \partial\mathbb{D} \times \mathbb{R}_+ \}, \\ \partial\mathbf{\Lambda} &= \{ (x, \theta, \eta) : x \in \mathbb{R}, \theta \in \partial\mathbb{D}, \eta = 0 \}. \end{aligned} \quad (4.1)$$

4.1 Local formulation of the scaled problem near boundary

Here is the local form of problem (2.21) in the neighborhood of $\partial\Gamma_\varepsilon$:

Lemma 4.1. *In terms of the coordinate system (s, θ, η) , the equation in (2.21) is expressed as*

$$u_{ss} + \frac{1}{l_1^2(\varepsilon\theta)}u_{\theta\theta} + \frac{1}{l_2^2(\varepsilon\theta)}u_{\eta\eta} - u + u^p + \tilde{B}(u) = 0, \quad (4.2)$$

where $\tilde{B}(u) = \tilde{B}_0(u) + \tilde{B}_1(u) + \tilde{B}_2(u)$ with components

$$\begin{aligned} \tilde{B}_0(u) = & \varepsilon \left[-2\eta \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)}u_{\eta\eta} + \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta)l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \right) u_\eta \right] - \varepsilon^2 s(k_1^2 + k_2^2)u_s \\ & + \varepsilon \left(\frac{E(\varepsilon\theta)}{l_1^2(\varepsilon\theta)l_2^2(\varepsilon\theta)} - \frac{R(\varepsilon\theta)}{l_1^4(\varepsilon\theta)} \right) u_\theta + \varepsilon^3 b(\varepsilon\theta, \varepsilon\eta)s^2u_s, \end{aligned}$$

$$\tilde{B}_1(u) = -\varepsilon \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)}u_\eta - \varepsilon \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)}u_\theta - 2\varepsilon s \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)}u_{s\eta} - 2\varepsilon s \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)}u_{s\theta},$$

$$\tilde{B}_2(u) = \varepsilon^2 a_1 u_\theta + \varepsilon^2 a_2 u_\eta + \varepsilon^2 b_1 u_{s\theta} + \varepsilon^2 b_2 u_{s\eta} + \tilde{B}_3(u),$$

we have defined for $\varepsilon\theta = \vartheta$

$$\begin{aligned} l_1(\vartheta) &= \left| \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta} \right| > 0, & l_2(\vartheta) &= \left| \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho} \right| > 0, \\ A(\vartheta) &= \left\langle \frac{\partial^2\gamma_0(\varepsilon\theta, 0)}{\partial\rho^2}, \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho} \right\rangle, & I(\vartheta) &= \left\langle q_1(\varepsilon\theta, 0), \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho} \right\rangle, \\ C(\vartheta) &= \left\langle \frac{\partial^2\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta\partial\rho}, \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta} \right\rangle, & R(\vartheta) &= \left\langle \frac{\partial^2\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta^2}, \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta} \right\rangle, \\ E(\vartheta) &= \left\langle \frac{\partial^2\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta\partial\rho}, \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho} \right\rangle, & F(\vartheta) &= \left\langle q_1(\varepsilon\theta, 0), \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta} \right\rangle, \end{aligned} \quad (4.3)$$

and the differential operator $\tilde{B}_3(s, \varepsilon\theta, \varepsilon\eta)$ is of size $O(\varepsilon^4)$. Here, a_1, a_2, b_1, b_2 are the coefficients of the operator B_2 in (2.28).

The boundary condition in (2.21) is recast as

$$\begin{aligned} \varepsilon I(\varepsilon\theta)su_s - u_\eta - \frac{2\varepsilon s}{l_2(\varepsilon\theta)^2} \left\langle \frac{\partial\nu}{\partial\rho}, \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho} \right\rangle u_\eta + \varepsilon \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)}su_\theta \\ + \varepsilon^2 \frac{I(\varepsilon\theta)}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial\nu}{\partial\rho}, \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho} \right\rangle s^2u_s + \varepsilon^2 b_4(\varepsilon\theta)s^2u_\eta \\ + \varepsilon^3 b_5(\varepsilon\theta)s^3u_s + \varepsilon^2 b_6(\varepsilon s, \varepsilon\theta)u_\theta + \tilde{D}(u) = 0, \end{aligned} \quad (4.4)$$

where we have denoted

$$M(\varepsilon\theta) = \left\langle \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\vartheta}, \frac{\partial\nu}{\partial\rho} \right\rangle + \left\langle \frac{\partial\gamma_0(\varepsilon\theta, 0)}{\partial\rho}, \frac{\partial\nu}{\partial\vartheta} \right\rangle.$$

The differential operator $\tilde{D}(\varepsilon s, \varepsilon\theta)$ is of size $O(\varepsilon^4)$ and functions $b_4(\varepsilon\theta)$, $b_5(\varepsilon\theta)$, $b_6(\varepsilon s, \varepsilon\theta)$ satisfy:

$$|b_4(\varepsilon\theta), b_5(\varepsilon\theta), b_6(\varepsilon s, \varepsilon\theta)| \leq C(1 + |\varepsilon s|^4).$$

Proof. The reader can refer to Section 4 in [34] and the references therein for the details of the proof. \square

As we have done in (3.33), we introduce a further changing of variables and define a new function $v(x, \theta, \eta)$ as follows

$$u(s, \theta, \eta) = v(x, \theta, \eta) \quad \text{with} \quad x = s - \sum_{i=0}^2 \varepsilon^i f_i(\varepsilon\theta, \varepsilon\eta), \quad \theta = \theta, \quad \eta = \eta. \quad (4.5)$$

We now want to express the problem in the new coordinates. Whence we need the following formulas

$$\begin{aligned} u_s &= v_x, & u_{ss} &= v_{xx}, \\ u_{s\theta} &= \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\vartheta} f_i \right) v_{xx} + v_{x\theta}, & u_{s\eta} &= \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right) v_{xx} + v_{x\eta}, \\ u_{\eta} &= \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right) v_x + v_{\eta}, & u_{\theta} &= \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\vartheta} f_i \right) v_x + v_{\theta}, \\ u_{\theta\theta} &= \left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\vartheta} f_i \right)^2 v_{xx} + \left(- \sum_{i=0}^2 \varepsilon^{i+2} \Delta_{\vartheta} f_i \right) v_x + \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\vartheta} f_i \right) v_{x\theta} + v_{\theta\theta}, \\ u_{\eta\eta} &= \left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right)^2 v_{xx} + \left(- \sum_{i=0}^2 \varepsilon^{i+2} \Delta_{\rho} f_i \right) v_x + \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right) v_{x\eta} + v_{\eta\eta}. \end{aligned}$$

Using Lemma 4.1, we get the local form of the problem (2.21) in the new coordinates and then extend it to the infinite strip $\mathbf{\Lambda}$

$$S(v) \equiv v_{xx} + \frac{1}{l_1^2(\varepsilon\theta)} v_{\theta\theta} + \frac{1}{l_2^2(\varepsilon\theta)} v_{\eta\eta} - v + v^p + \tilde{B}(v) = 0 \quad \text{in } \mathbf{\Lambda}, \quad (4.6)$$

where we have denoted

$$\tilde{B}(v) = \tilde{B}_0(v) + \tilde{B}_1(v) + \tilde{B}_2(v), \quad (4.7)$$

$$\begin{aligned} \tilde{B}_0(v) &= \frac{1}{l_2^2(\varepsilon\theta)} \left[\left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right)^2 v_{xx} + \left(- \sum_{i=0}^2 \varepsilon^{i+2} \Delta_{\rho} f_i \right) v_x + \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right) v_{x\eta} \right] \\ &\quad - 2\varepsilon\eta \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \left[\left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right)^2 v_{xx} + \left(- \sum_{i=0}^2 \varepsilon^{i+2} \Delta_{\rho} f_i \right) v_x + \left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right) v_{x\eta} + v_{\eta\eta} \right] \\ &\quad + \varepsilon \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \right) \left[\left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\rho} f_i \right) v_x + v_{\eta} \right] - \varepsilon^2 \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) (k_1^2 + k_2^2) v_x \\ &\quad + \varepsilon^2 \frac{1}{l_1^2(\varepsilon\theta)} \left[\left(\sum_{i=0}^2 \varepsilon^i \nabla_{\vartheta} f_i \right)^2 v_{xx} + \left(- \sum_{i=0}^2 \varepsilon^i \Delta_{\vartheta} f_i \right) v_x + \left(- \sum_{i=0}^2 \varepsilon^i \nabla_{\vartheta} f_i \right) v_{x\theta} \right] \\ &\quad + \varepsilon \left(\frac{E(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{R(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \right) \left[\left(- \sum_{i=0}^2 \varepsilon^{i+1} \nabla_{\vartheta} f_i \right) v_x + v_{\theta} \right] + \varepsilon^3 b(\varepsilon\theta, \varepsilon\eta) \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) v_x, \end{aligned}$$

$$\begin{aligned}
\tilde{B}_1(v) &= -\varepsilon \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_x + v_\eta \right] - \varepsilon \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\vartheta f_i \right) v_x + v_\theta \right] \\
&\quad - 2\varepsilon \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_{xx} + v_{x\eta} \right] \\
&\quad - 2\varepsilon \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\vartheta f_i \right) v_{xx} + v_{x\theta} \right], \\
\tilde{B}_2(v) &= \varepsilon^2 a_1 \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\vartheta f_i \right) v_x + v_\theta \right] + \varepsilon^2 a_2 \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_x + v_\eta \right] \\
&\quad + \varepsilon^2 b_1 \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\vartheta f_i \right) v_{xx} + v_{x\theta} \right] + \varepsilon^2 b_2 \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_{xx} + v_{x\eta} \right] + \tilde{B}_3(v).
\end{aligned}$$

The boundary condition is

$$\begin{aligned}
&\varepsilon \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) I(\varepsilon\theta) v_x + \left(\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_x - v_\eta \\
&\quad - \frac{2\varepsilon}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0(\varepsilon\theta, 0)}{\partial \rho} \right\rangle \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_x + v_\eta \right] \\
&\quad + \varepsilon^2 \frac{I(\varepsilon\theta)}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0(\varepsilon\theta, 0)}{\partial \rho} \right\rangle \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right)^2 v_x + \varepsilon^3 b_5(\varepsilon\theta) \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right)^3 v_x \\
&\quad + \varepsilon^2 b_4(\varepsilon\theta) \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right)^2 \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) v_x + v_\eta \right] \\
&\quad + \varepsilon \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \left(x + \sum_{i=0}^2 \varepsilon^i f_i \right) \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\vartheta f_i \right) v_x + v_\theta \right] \\
&\quad + \varepsilon^2 b_6(x, \varepsilon\theta) \left[\left(-\sum_{i=0}^2 \varepsilon^{i+1} \nabla_\vartheta f_i \right) v_x + v_\theta \right] + \tilde{D}(v) = 0 \quad \text{on } \partial\Lambda.
\end{aligned} \tag{4.8}$$

4.2 Boundary correction layers

In this subsection, we will find some boundary correction layer terms, say ϕ_i 's, by also the recurrence method. The method is basically the same as that in [39] and [40].

By recalling (3.36), we take

$$u_1 \equiv \mathcal{V} = w(x) + \varepsilon eZ + \sum_{i=2}^3 \varepsilon^i \varphi_i, \tag{4.9}$$

as the first approximate solution of the problem (4.6) and (4.8) on Λ . Then we compute

$$S(u_1) = \varepsilon^3 \Delta^\Gamma eZ + \varepsilon \lambda_0 eZ + \tilde{B}_3(u_1) \quad \text{in } \Lambda. \tag{4.10}$$

On the boundary, the errors become

$$\Phi \equiv \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + \varepsilon^3 \Phi_3 + \tilde{D}(u_1) = 0 \quad \text{on } \partial\Lambda,$$

where we have denoted

$$\begin{aligned}
\Phi_1 &= \left(I(\varepsilon\theta)x + I(\varepsilon\theta)f_0 + \nabla_\rho f_0 \right) w_x, \\
\Phi_2 &= \nabla_\rho f_1 w_x + I(\varepsilon\theta)f_1 w_x - \nabla_\rho e Z + \nabla_\rho f_0 e Z_x \\
&\quad + I(\varepsilon\theta) \left(x + f_0 \right) e Z_x + \frac{2}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0}{\partial \rho} \right\rangle \left(x + f_0 \right) \nabla_\rho f_0 w_x \\
&\quad + \frac{I(\varepsilon\theta)}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0(\varepsilon\theta, 0)}{\partial \rho} \right\rangle \left(x + f_0 \right)^2 w_x - \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \left(x + f_0 \right) \nabla_\vartheta f_0 w_x, \\
\Phi_3 &= \nabla_\rho f_2 w_x + I(\varepsilon\theta)f_2 w_x + \nabla_\rho f_1 e Z_x + I(\varepsilon\theta)f_1 e Z_x - \varphi_{2,\eta} + \varepsilon^3 b_5(\varepsilon\theta) \left(x + f_0 \right)^3 w_x \\
&\quad + \frac{2}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0}{\partial \rho} \right\rangle \left[\left(x + f_0 \right) \left(\nabla_\rho f_1 w_x + \nabla_\rho f_0 e Z_x - \nabla_\rho e Z \right) + f_1 \nabla_\rho f_0 w_x \right] \\
&\quad + \frac{I(\varepsilon\theta)}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0(\varepsilon\theta, 0)}{\partial \rho} \right\rangle \left[\left(x + f_0 \right)^2 e Z_x + 2f_1 x w_x \right] - b_4(\varepsilon\theta) \left(x + f_0 \right)^2 \nabla_\rho f_0 w_x \\
&\quad - \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \left[\left(x + f_0 \right) \left(\nabla_\vartheta f_1 w_x + \nabla_\vartheta f_0 e Z_x - \nabla_\vartheta e Z \right) + f_1 \nabla_\vartheta f_0 w_x \right] \\
&\quad - b_6(x, \varepsilon\theta) \nabla_\vartheta f_0 w_x.
\end{aligned}$$

To improve the approximation, we need to cancel the terms Φ_i 's by adding boundary correction terms to the inner approximate expansion in (4.9). This can be done step by step. On the boundary it is natural to take

$$\nabla_\rho f_0 + I(\varepsilon\theta)f_0 = 0, \quad (4.11)$$

which will lead to the cancelation of $(I(\varepsilon\theta)f_0 + \nabla_\rho f_0)w_x$ in the components of the error of first order of ε . Recall that by combining of (3.59) and (4.11) we have chosen $f_0 \equiv 0$ in Section 3. On the other hand, to cancel the first order term like $\varepsilon I(\varepsilon\theta)xw_x$ on the boundary, we shall introduce a boundary layer term, say ϕ_1 . Indeed, we first introduce the term

$$\tilde{\phi}(x, \theta, \eta) = \varepsilon A_0(\varepsilon\theta) b(\sqrt{\lambda_0} l_2(\varepsilon\theta)\eta) Z(x),$$

with $\tilde{\phi}(x, \theta, \eta)$ satisfying

$$\begin{aligned}
\tilde{\phi}_{xx} + \frac{1}{l_1^2(\varepsilon\theta)} \tilde{\phi}_{\theta\theta} + \frac{1}{l_2^2(\varepsilon\theta)} \tilde{\phi}_{\eta\eta} - \tilde{\phi} + pw^{p-1}\tilde{\phi} &= O(\varepsilon^3) \quad \text{in } \mathbf{\Lambda}, \\
\frac{\partial \tilde{\phi}}{\partial \eta} &= -c_0(\varepsilon\theta)Z, \quad \text{on } \partial\mathbf{\Lambda},
\end{aligned}$$

where $c_0(\varepsilon\theta)$ is a function in the parameter θ of the form

$$c_0(\varepsilon\theta) = I(\varepsilon\theta) \int_{\mathbb{R}} x w_x Z \, dx. \quad (4.12)$$

In fact, we can choose

$$\tilde{\phi}(x, \theta, \eta) = \varepsilon \frac{-c_0(\varepsilon\theta)}{\sqrt{\lambda_0} l_2(\varepsilon\theta)} \sin(\sqrt{\lambda_0} l_2(\varepsilon\theta)\eta) Z(x) \equiv \varepsilon \phi_{11}(x, \theta, \eta). \quad (4.13)$$

Then, by Corollary A.1.1, there exists a unique solution (denoted by ϕ_{12}) of the following problem

$$\begin{aligned} \phi_{12,xx} + \frac{1}{l_1^2(\varepsilon\theta)} \phi_{12,\theta\theta} + \frac{1}{l_2^2(\varepsilon\theta)} \phi_{12,\eta\eta} - \phi_{12} + pw^{p-1} \phi_{12} &= 0 \quad \text{in } \mathbf{\Lambda}, \\ \frac{\partial \phi_{12}}{\partial \eta} &= I(\varepsilon\theta) x w_x - c_0(\varepsilon\theta) Z \quad \text{on } \partial \mathbf{\Lambda}. \end{aligned}$$

Moreover, ϕ_{12} is even in the variable x . By choosing a smooth cut-off function χ_0 in the form

$$\chi_0(t) = 1 \quad \text{if } |t| < 1, \quad \chi_0(t) = 0 \quad \text{if } |t| > 2, \quad (4.14)$$

we can set the **first boundary layer term** by

$$\phi_1 = \varepsilon \chi_0(\varepsilon\eta) \Psi_1(x, \theta, \eta) \quad \text{with} \quad \Psi_1(x, \theta, \eta) = \phi_{11}(x, \theta, \eta) + \phi_{12}(x, \theta, \eta). \quad (4.15)$$

Thus we finish the first step.

To proceed the second step, let $u_2 = u_1 + \phi_1$ be the second approximate solution. We again compute the new error

$$\begin{aligned} S(u_2) &= S(u_1) + 2\varepsilon^2 \frac{1}{l_2^2(\varepsilon\theta)} \chi_0' \Psi_{1,\eta} + \varepsilon^3 \frac{1}{l_2^2(\varepsilon\theta)} \chi_0'' \Psi_1 + \frac{p(p-1)}{2} w^{p-2} (\varepsilon e Z + \phi_1)^2 \\ &\quad + \varepsilon^2 \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{1,\eta} \\ &\quad - \varepsilon^2 \left(\frac{1}{l_2^2(\varepsilon\theta)} + 2x \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{1,x\eta} \\ &\quad + \mathcal{C}_0(\phi_{11}) + L_0(\phi_1) + N_0(\phi_1) \quad \text{in } \mathbf{\Lambda}, \end{aligned} \quad (4.16)$$

where $\mathcal{C}_0(\phi_{11})$, $L_0(\phi_1)$ and $N_0(\phi_1)$ are of size $O(\varepsilon^3)$ and $S(u_1)$ is defined in (4.10),

$$\begin{aligned} \mathcal{C}_0(\phi_{11}) &= S(\phi_{11}) + \varepsilon^2 \left(\frac{E(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{R(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \phi_{11,\theta} \\ &\quad - 2\varepsilon^2 x \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \phi_{11,x\theta}, \\ L_0(\phi_1) &= \tilde{B}(\phi_1) - \varepsilon^2 \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{1,\eta} \\ &\quad + \varepsilon^2 \left(\frac{1}{l_2^2(\varepsilon\theta)} + 2x \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{1,x\eta}, \\ N_0(\phi_1) &= (u_1 + \phi_1)^p - u_1^p - pw^{p-1} \phi_1 - \frac{p(p-1)}{2} w^{p-2} (\varepsilon e Z + \phi_1)^2. \end{aligned}$$

Note that $f_0 \equiv 0$. On the boundary, the new error becomes

$$\begin{aligned}
& \varepsilon^2 \left(\nabla_\rho f_1 w_x + I(\varepsilon\theta) f_1 w_x + I(\varepsilon\theta) e x Z_x - \nabla_\rho e Z \right) + \varepsilon^2 I(\varepsilon\theta) x \phi_{12,x} \\
& + \varepsilon^2 \frac{I(\varepsilon\theta)}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0(\varepsilon\theta, 0)}{\partial \rho} \right\rangle x^2 w_x + \varepsilon^2 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \phi_{12,\theta} \\
& + \varepsilon^3 \left[\nabla_\rho f_2 w_x + I(\varepsilon\theta) f_2 w_x + \nabla_\rho f_1 (e Z_x + \phi_{12,x}) + I(\varepsilon\theta) f_1 (e Z_x + \phi_{12,x}) - \varphi_{2,\eta} \right] \\
& + \varepsilon^3 \frac{2x}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0}{\partial \rho} \right\rangle (\nabla_\rho f_1 w_x - \nabla_\rho e Z) \\
& + \varepsilon^3 \frac{I(\varepsilon\theta)}{l_2^2(\varepsilon\theta)} \left\langle \frac{\partial \nu}{\partial \rho}, \frac{\partial \gamma_0(\varepsilon\theta, 0)}{\partial \rho} \right\rangle \left[x^2 (e Z_x + \phi_{12,x}) + 2 f_1 x w_x \right] \\
& + \varepsilon^3 b_4(\varepsilon\theta) x^2 \Psi_{1,\eta} + \varepsilon^3 b_5(\varepsilon\theta) x^3 w_x - \varepsilon^3 b_6(x, \varepsilon\theta) \nabla_\vartheta f_0 w_x \\
& - \varepsilon^3 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} x (\nabla_\vartheta f_1 w_x - \nabla_\vartheta e Z) + \tilde{D}(u_2) = 0 \quad \text{on } \partial \mathbf{\Lambda}.
\end{aligned}$$

In order to cancel the error terms of order $O(\varepsilon^2)$ on the boundary, we do the same thing as that in the first step. We first set

$$\nabla_\rho f_1 + I(\varepsilon\theta) f_1 = 0, \quad (4.17)$$

and

$$- \int_{\mathbb{R}} \left[I(\varepsilon\theta) e x Z_x - \nabla_\rho e Z \right] Z \, dx = \nabla_\rho e + \frac{1}{2} I(\varepsilon\theta) e = 0. \quad (4.18)$$

Then we choose

$$\hat{\phi}(x, \theta, \eta) = \varepsilon^2 \frac{-c_1(\varepsilon\theta) - c_2(\varepsilon\theta)}{\sqrt{\lambda_0} l_2(\varepsilon\theta)} \sin(\sqrt{\lambda_0} l_2(\varepsilon\theta) \eta) Z(x) \equiv \varepsilon^2 \phi_{21}(x, \theta, \eta), \quad (4.19)$$

where $c_1(\varepsilon\theta)$ and $c_2(\varepsilon\theta)$ are functions in the parameter θ of the form

$$c_1(\varepsilon\theta) = I(\varepsilon\theta) \int_{\mathbb{R}} x \phi_{12,x} Z \, dx, \quad c_2(\varepsilon\theta) = -\frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \int_{\mathbb{R}} \phi_{12,\theta} Z \, dx. \quad (4.20)$$

Now, by Lemma A.1.3, there exists a unique solution (denoted by ϕ_{22}) of the following problem

$$\phi_{22,xx} + \frac{1}{l_1^2(\varepsilon\theta)} \phi_{22,\theta\theta} + \frac{1}{l_2^2(\varepsilon\theta)} \phi_{22,\eta\eta} - \phi_{22} + p w^{p-1} \phi_{22} = h_1 \quad \text{in } \mathbf{\Lambda},$$

$$\frac{\partial \phi_{22}}{\partial \eta} = I(\varepsilon\theta) e x Z_x - \nabla_\rho e Z + I(\varepsilon\theta) x \phi_{12,x} - c_1(\varepsilon\theta) Z - \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \phi_{12,\theta} - c_2(\varepsilon\theta) Z \quad \text{on } \partial \mathbf{\Lambda},$$

where h_1 is a function in the form

$$\begin{aligned}
h_1 = & -2 \frac{1}{l_2^2(\varepsilon\theta)} \chi_0' \Psi_{1,\eta} - \frac{p(p-1)}{2} w^{p-2} (\varepsilon e Z + \phi_1)^2 \\
& + \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{1,\eta} - \left(\frac{1}{l_2^2(\varepsilon\theta)} + 2x \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{1,x\eta}.
\end{aligned}$$

Moreover ϕ_{22} is even in the variable x . By recalling the cut-off function in (4.14), we set the **second boundary layer term** by

$$\phi_2 = \varepsilon^2 \chi_0(\varepsilon\eta) \Psi_2(x, \theta, \eta) \quad \text{with} \quad \Psi_2(x, \theta, \eta) = \phi_{21}(x, \theta, \eta) + \phi_{22}(x, \theta, \eta). \quad (4.21)$$

For the completeness, we here give the third step although it is the same as we have done in the above. Let $u_3 = u_2 + \phi_2$ be the second approximate solution. We compute the new error

$$\begin{aligned} S(u_3) = & S(u_2) + \varepsilon^3 \frac{1}{l_2^2(\varepsilon\theta)} \chi_0'' \Psi_1 + 2\varepsilon^3 \frac{1}{l_2^2(\varepsilon\theta)} \chi_0' \Psi_{2,\eta} + \varepsilon^4 \frac{1}{l_2^2(\varepsilon\theta)} \chi_0'' \Psi_2 \\ & + p(p-1)w^{p-2}\varepsilon eZ\phi_2 + \varepsilon^3 \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta)l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{2,\eta} \\ & - \varepsilon^3 \left(\frac{1}{l_2^2(\varepsilon\theta)} + 2x \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{2,x\eta} - 4\varepsilon^3 \eta \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \chi_0' \Psi_{1,\eta} \\ & - \varepsilon^3 x(k_1^2 + k_2^2) \chi_0 \Psi_{1,x} + \mathcal{C}_0 + \mathcal{C}_1 + L_1(\phi_2) + N_1(\phi_2) \quad \text{in } \Lambda, \end{aligned}$$

where $\mathcal{C}_1(\phi_{21})$, $L_0(\phi_2)$ and $N_0(\phi_2)$ are of size $O(\varepsilon^4)$ and $S(u_2)$ is defined in (4.16),

$$\begin{aligned} \mathcal{C}_1(\phi_{21}) = & \varepsilon S(\phi_{21}) + \varepsilon^3 \left(\frac{E(\varepsilon\theta)}{l_1^2(\varepsilon\theta)l_2^2(\varepsilon\theta)} - \frac{R(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \right) \phi_{21,\theta} \\ & - 2\varepsilon^3 x \frac{F(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \phi_{21,x\theta}, \\ L_1(\phi_2) = & \tilde{B}(\phi_2) + L_0(\phi_1) - \varepsilon^3 \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta)l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{2,\eta} \\ & + \varepsilon^3 \left(\frac{1}{l_2^2(\varepsilon\theta)} + 2x \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta)l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{2,x\eta} \\ & + 4\varepsilon^3 \eta \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \chi_0' \Psi_{1,\eta} - \varepsilon^4 x(k_1^2 + k_2^2) \chi_0 \Psi_{2,x}, \\ N_1(\phi_2) = & (u_2 + \phi_2)^p - u_2^p - pw^{p-1}\phi_2 - p(p-1)w^{p-2}\varepsilon eZ\phi_2. \end{aligned}$$

On the boundary, the new error becomes

$$\begin{aligned} & \varepsilon^3 (\nabla_\rho f_2 w_x + I(\varepsilon\theta) f_2 w_x + I(\varepsilon\theta) x \varphi_{2,x} - \varphi_{2,\eta}) \\ & + \varepsilon^3 I(\varepsilon\theta) x \phi_{22,x} + \varepsilon^3 b_4(\varepsilon\theta) x^2 \Psi_{1,\eta} + \varepsilon^3 b_5(\varepsilon\theta) x^3 w_x \\ & - \varepsilon^3 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} x (\nabla_\vartheta f_1 w_x - \nabla_\vartheta eZ) + \tilde{D}(u_3) = 0 \quad \text{on } \partial\Lambda. \end{aligned}$$

In order to cancel the error terms of order $O(\varepsilon^3)$ on the boundary, we first set

$$\nabla_\rho f_2 + I(\varepsilon\theta) f_2 + \mathfrak{R}(\theta) = 0, \quad (4.22)$$

by introducing the term

$$\mathfrak{R}(\theta) = \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \nabla_\vartheta e \int_{\mathbb{R}} x Z w_x dx \Big/ \int_{\mathbb{R}} w_x^2 dx. \quad (4.23)$$

Then we also introduce the term of the form

$$\check{\phi}(x, \theta, \eta) = \varepsilon \frac{-c_3(\varepsilon\theta)}{\sqrt{\lambda_0} l_2(\varepsilon\theta)} \sin\left(\sqrt{\lambda_0} l_2(\varepsilon\theta) \eta\right) Z(x) \equiv \varepsilon \phi_{31}(x, \theta, \eta), \quad (4.24)$$

where $c_3(\varepsilon\theta)$ is a function of the form

$$c_3(\varepsilon\theta) = \int_{\mathbb{R}} c_4(x, \varepsilon\theta) Z \, dx. \quad (4.25)$$

In the above, we have denoted

$$\begin{aligned} c_4(x, \varepsilon\theta) = & I(\varepsilon\theta)x \varphi_{2,x} + I(\varepsilon\theta)x \phi_{22,x} - \varphi_{2,\eta} + b_4(\varepsilon\theta)x^2 \\ & + b_5(\varepsilon\theta)x^3 - \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \nabla_{\vartheta} f_1 x w_x, \end{aligned} \quad (4.26)$$

Again, by Lemma A.1.3, there exists a unique solution (denoted by ϕ_{32}) of the following problem

$$\begin{aligned} \phi_{32,xx} + \frac{1}{l_1^2(\varepsilon\theta)} \phi_{32,\theta\theta} + \frac{1}{l_2^2(\varepsilon\theta)} \phi_{32,\eta\eta} - \phi_{32} + p w^{p-1} \phi_{32} &= h_2 \quad \text{in } \Lambda, \\ \frac{\partial \phi_{32}}{\partial \eta} &= c_4(x, \varepsilon\theta) - c_3(\varepsilon\theta) Z \quad \text{on } \partial \Lambda, \end{aligned}$$

where h_2 is a function of the form

$$\begin{aligned} h_2 = & - \left(\frac{C(\varepsilon\theta)}{l_1^2(\varepsilon\theta) l_2^2(\varepsilon\theta)} - \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} - \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{2,\eta} - p(p-1) w^{p-2} \varepsilon e Z \phi_2 \\ & + \left(\frac{1}{l_2^2(\varepsilon\theta)} + 2x \frac{I(\varepsilon\theta)}{l_1(\varepsilon\theta) l_2(\varepsilon\theta)} \right) \chi_0 \Psi_{2,x\eta} + 4\eta \frac{A(\varepsilon\theta)}{l_2^4(\varepsilon\theta)} \chi_0' \Psi_{1,\eta} - \frac{1}{l_2^2(\varepsilon\theta)} \chi_0'' \Psi_1 \\ & + \mathcal{C}_0/\varepsilon^3 + x(k_1^2 + k_2^2) \chi_0 \Psi_{1,x}. \end{aligned}$$

Moreover ϕ_{22} is even in the variable x . We can set the **third boundary layer term** by

$$\phi_3 = \varepsilon^3 \chi_0(\varepsilon\eta) \Psi_3(x, \theta, \eta) \quad \text{with} \quad \Psi_3(x, \theta, \eta) = \phi_{31}(x, \theta, \eta) + \phi_{32}(x, \theta, \eta). \quad (4.27)$$

Remark 4.2. *In fact, we can proceed the above arguments step by step to find boundary correction terms and get rid of the error terms up to order $O(\varepsilon^m)$ for any positive integer m .*

4.3 Summary

We conclude that for any given parameter pair $(f_2, e) \in F$, our **final approximate solution** to the problem (2.21) near the surface Γ_ε is expressed in the local form by

$$u_4 = w(x) + \varepsilon e Z + \sum_{i=2}^3 \varepsilon^i \varphi_i + \phi_1 + \phi_2 + \phi_3. \quad (4.28)$$

For a suitable perturbation term ϕ , if we locally set $u_4 + \phi$ as the solution to problem (2.21), the problem can be recast as follows

$$S(u_4 + \phi) = S(u_4) + L_2(\phi) + B(\phi) + N_2(\phi) = 0, \quad (4.29)$$

where the linear and nonlinear operators are in the form

$$\begin{aligned} L_2(\phi) &= \phi_{xx} + \Delta^{\Gamma_\varepsilon} \phi - \phi + pu_4^{p-1} \phi, \\ N_2(\phi) &= (u_4 + \phi)^p - u_4^p - pu_4^{p-1} \phi, \end{aligned} \quad (4.30)$$

with boundary condition

$$\tilde{D}_0(\phi) - \phi_\eta + \tilde{D}(u_4 + \phi) = g. \quad (4.31)$$

The error of the approximation is

$$\begin{aligned} E_1 &= S(u_4) \\ &= \varepsilon^3 \Delta^\Gamma eZ + \varepsilon \lambda_0 eZ + \varepsilon^4 (\hat{\mathbf{B}}_4 + \check{\mathbf{B}}_4 + \hat{G}_4 + \check{G}_4 + \hat{\mathfrak{D}}_4 + \check{\mathfrak{D}}_4) \\ &\quad + \varepsilon^4 p(p-1)w^{p-2}eZ\phi_3 + \sum_{i=4}^6 \varepsilon^i S_i + \sum_{i=4}^6 \varepsilon^i T_i + \sum_{i=5}^{10} \varepsilon^i G_i + \varepsilon^5 \mathbf{B}_5. \end{aligned} \quad (4.32)$$

In the above, we also have denoted

$$\begin{aligned} \tilde{D}_0(\phi) &= \varepsilon I(\varepsilon\theta)x\phi_x + \varepsilon^2 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} x\phi_\theta + \varepsilon^3 b_5(\varepsilon\theta)x^3\phi_x \\ &\quad + \varepsilon^3 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} f_1\phi_\theta + \varepsilon^3 b_4(\varepsilon\theta)x^2\phi_\eta, \end{aligned} \quad (4.33)$$

$$\begin{aligned} g(x, \theta) &= \varepsilon^4 I(\varepsilon\theta)x\varphi_{3,x} + \varepsilon^4 I(\varepsilon\theta)x\phi_{32,x} + \sum_{i=1}^2 \varepsilon^{i+3} \left(f_i I(\varepsilon\theta) + \nabla_\rho f_i \right) \varphi_{i+2,x} \\ &\quad + \varepsilon^2 b_4(\varepsilon\theta) \left(x + \sum_{i=1}^2 \varepsilon^i f_i \right)^2 \left(\varepsilon^2 \nabla_\rho eZ + \sum_{i=2}^4 \varepsilon^i \varphi_{i,\eta} \right) - \varepsilon^4 \varphi_{3,\eta} \\ &\quad - \varepsilon^2 b_4(\varepsilon\theta) \left(x + \sum_{i=1}^2 \varepsilon^i f_i \right)^2 \left(\sum_{i=1}^2 \varepsilon^{i+1} \nabla_\rho f_i \right) \left(\varepsilon eZ_x + \sum_{i=2}^4 \varepsilon^i \varphi_{i,x} \right) \\ &\quad - \varepsilon^2 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \left(x + \sum_{i=1}^2 \varepsilon^i f_i \right) \left(\sum_{i=1}^2 \varepsilon^i \nabla_\vartheta f_i \right) \left(\sum_{i=2}^4 \varepsilon^i \varphi_{i,x} + \sum_{i=2}^3 \varepsilon^i \phi_{i2,x} \right) \\ &\quad + \varepsilon^2 \frac{M(\varepsilon\theta)}{l_1^2(\varepsilon\theta)} \left(x + \sum_{i=1}^2 \varepsilon^i f_i \right) \left(\sum_{i=2}^4 \varepsilon^i \varphi_{i,\theta} + \sum_{i=2}^3 \varepsilon^i \phi_{i2,\theta} \right) \\ &\quad + \varepsilon^3 b_6(x, \varepsilon\theta) \left(\varepsilon \nabla_\vartheta eZ + \sum_{i=2}^4 \varepsilon^i \varphi_{i,\theta} + \sum_{i=1}^3 \varepsilon^i \phi_{i2,\theta} \right) \\ &\quad + \varepsilon^3 b_6(x, \varepsilon\theta) \left(- \sum_{i=1}^2 \varepsilon^i \nabla_\vartheta f_i \right) \left(\varepsilon eZ_x + \sum_{i=2}^4 \varepsilon^i \varphi_{i,x} + \sum_{i=1}^3 \varepsilon^i \phi_{i2,x} \right). \end{aligned} \quad (4.34)$$

For later use, we decompose the error as two components

$$E_1 = E_{11} + E_{12}, \quad (4.35)$$

where we have denoted

$$E_{11} = \varepsilon^3 \Delta^\Gamma eZ + \varepsilon \lambda_0 eZ \quad \text{and} \quad E_{12} = E_1 - E_{11}.$$

4.4 Size of the errors in weighted Sobolev norms

To estimate the size of error, we have to introduce some suitable weighted Sobolev norms. Here we use the same norms as those in [12]: for a function $h(x, z)$ defined on a set $\mathbb{E} \in \mathbb{R}^3$, and for $0 < \varrho < \frac{1}{100}$ and $4 < q \leq +\infty$, we set

$$\begin{aligned} \|h\|_{q, \varrho; \mathbb{E}} &= \sup_{(x, z) \in \mathbb{E}} e^{\varrho|x|} \|h\|_{L^q(B((x, z), 1))}, \\ \|h\|_{2, q, \varrho; \mathbb{E}} &= \sum_{j=0}^2 \sup_{(x, z) \in \mathbb{E}} e^{\varrho|x|} \|D^j h\|_{L^q(B((x, z), 1))}. \end{aligned} \quad (4.36)$$

Here $B((x, z), 1)$ denotes the ball of radius 1 centered at (x, z) .

For the application of contraction mapping theorem in the procedure of finding the perturbation term ϕ , we need analyze the properties of g , E_{11} and E_{12} . The reader can refer to Section 7. From the uniform bound of e in (3.37), it is easy to see that

$$\|E_{11}\|_{q, \varrho; \mathfrak{S}} \leq C\varepsilon^{\frac{1}{2}+1-\frac{3}{q}}. \quad (4.37)$$

where \mathfrak{S} is defined in (3.29).

All terms in E_{12} carry ε^4 in front, we then claim that

$$\|E_{12}\|_{q, \varrho; \mathfrak{S}} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}}. \quad (4.38)$$

A rather delicate term in E_{12} is the one carrying $\Delta^\Gamma f_2$ since we only assume a uniform bound on $\|\Delta^\Gamma f_2\|_{L^q(\Gamma)}$. For example, we have a term $K_1 = \varepsilon^4 \Delta^\Gamma f_2 w_x$ in $S(w)$ which has bound like

$$\|K_1\|_{q, \varrho; \mathfrak{S}} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}}.$$

Similarly, we have the following estimates

$$\|g\|_{q, \varrho; \partial \mathfrak{S}} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}}. \quad (4.39)$$

Other terms can be estimated in the similar way. Moreover, for the Lipschitz dependence of the term of error E_{12} on the parameters f_2 and e for the norms defined in (3.32) and (3.37), we have the validity of the estimate

$$\|E_{12}(f_2, e) - E_{12}(\tilde{f}_2, \tilde{e})\|_{q, \varrho; \mathfrak{S}} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}} (\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b). \quad (4.40)$$

Similarly we obtain

$$\|g(f_2, e) - g(\tilde{f}_2, \tilde{e})\|_{q, \varrho; \partial \mathfrak{S}} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}} (\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b). \quad (4.41)$$

5 The Gluing Procedure

Recall that, in Sections 3 and 4, we consider problem (2.21) in a small neighborhood of Γ_ε and find a local approximate solution. In this section, to get a real solution to (2.21) by the perturbation method, we use a gluing technique (as in [10]) to reduce the problem in Ω_ε to a projected problem on the infinite strip \mathfrak{S} (cf. (3.29)) in \mathbb{R}^3 .

Let $\sigma < r_0/100$ be a fixed number, where r_0 is a constant defined in (2.7). We consider a smooth cut-off function $\eta_\sigma(t)$ where $t \in \mathbb{R}_+$ such that $\eta_\sigma(t) = 1$ for $0 \leq t \leq \sigma$ and $\eta(t) = 0$ for $t > 2\sigma$. Set $\eta_\sigma^\varepsilon(s) = \eta_\sigma(\varepsilon|s|)$, where s is the normal coordinate to Γ_ε . Let $u_4(s, z)$ denote the approximate solution constructed near the surface Γ_ε in the coordinates (s, z) . We define our first global approximation to be simply

$$W = \eta_{3\sigma}^\varepsilon(s)u_4. \quad (5.1)$$

Obviously, W is a function defined on Ω_ε , which is extended globally as 0 outside the $6\sigma/\varepsilon$ -neighborhood of Γ_ε .

For $u = W + \hat{\phi}$ where $\hat{\phi}$ globally defined in Ω_ε , denote

$$S(u) = \Delta_{\tilde{y}}u - u + u^p \quad \text{in } \Omega_\varepsilon.$$

Then u satisfies (1.1) if and only if

$$\tilde{\mathcal{L}}(\hat{\phi}) = -\tilde{E} - \tilde{N}(\hat{\phi}) \quad \text{in } \Omega_\varepsilon, \quad (5.2)$$

with boundary condition

$$\frac{\partial \hat{\phi}}{\partial \mathbf{n}_\varepsilon} + \frac{\partial W}{\partial \mathbf{n}_\varepsilon} = 0 \quad \text{on } \partial\Omega_\varepsilon, \quad (5.3)$$

where

$$\begin{aligned} \tilde{E} &= S(W), \quad \tilde{\mathcal{L}}(\hat{\phi}) = \Delta_{\tilde{y}}\hat{\phi} - \hat{\phi} + pW^{p-1}\hat{\phi}, \\ \tilde{N}(\hat{\phi}) &= (W + \hat{\phi})^p - W^p - pW^{p-1}\hat{\phi}. \end{aligned}$$

We will look for $\hat{\phi}$ in the following form

$$\hat{\phi} = \eta_{3\delta}^\varepsilon\phi + \psi,$$

where, in the coordinates (x, z) of the form (3.33), we assume that ϕ is defined in the whole strip \mathfrak{S} . Obviously, (5.2)-(5.3) is equivalent to the following problem

$$\eta_{3\delta}^\varepsilon\left(\Delta_{\tilde{y}}\phi - \phi + pW^{p-1}\phi\right) = \eta_\delta^\varepsilon\left[-\tilde{N}(\eta_{3\delta}^\varepsilon\phi + \psi) - \tilde{E} - pW^{p-1}\psi\right], \quad (5.4)$$

$$\begin{aligned} \Delta_{\tilde{y}}\psi - \psi + (1 - \eta_\delta^\varepsilon)pW^{p-1}\psi &= -\varepsilon^2(\Delta_{\tilde{y}}\eta_{3\delta}^\varepsilon)\phi - 2\varepsilon(\nabla_{\tilde{y}}\eta_{3\delta}^\varepsilon)(\nabla_{\tilde{y}}\phi) \\ &\quad - (1 - \eta_\delta^\varepsilon)\tilde{N}(\eta_{3\delta}^\varepsilon\phi + \psi) - (1 - \eta_\delta^\varepsilon)\tilde{E}. \end{aligned} \quad (5.5)$$

On the boundary, we get

$$\eta_{3\delta}^\varepsilon \frac{\partial \phi}{\partial \mathbf{n}_\varepsilon} + \eta_\delta^\varepsilon \frac{\partial W}{\partial \mathbf{n}_\varepsilon} = 0, \quad (5.6)$$

$$\frac{\partial \psi}{\partial \mathbf{n}_\varepsilon} + (1 - \eta_\delta^\varepsilon) \frac{\partial W}{\partial \mathbf{n}_\varepsilon} + \varepsilon \frac{\partial \eta_{3\delta}^\varepsilon}{\partial \mathbf{n}_\varepsilon} \phi = 0. \quad (5.7)$$

The key observation is that, after solving (5.5) and (5.7), the problem can be transformed to the following nonlinear problem involving the parameter ψ

$$\tilde{\mathcal{L}}(\phi) = \eta_\delta^\varepsilon \left[-\tilde{N}(\eta_{3\delta}^\varepsilon\phi + \psi) - \tilde{E} - pW^{p-1}\psi \right] \quad \text{in } \mathfrak{S}, \quad (5.8)$$

$$\frac{\partial \phi}{\partial \mathbf{n}_\varepsilon} + \eta_\delta^\varepsilon \frac{\partial W}{\partial \mathbf{n}_\varepsilon} = 0 \quad \text{on } \partial \mathfrak{S}. \quad (5.9)$$

Notice that the operator $\tilde{\mathcal{L}}$ in Ω_ε may be taken as any compatible extension outside the $6\sigma/\varepsilon$ -neighborhood of Γ/ε in the strip \mathfrak{S} and the operator $\partial/\partial \mathbf{n}_\varepsilon$ may be taken as any compatible extension outside the $6\sigma/\varepsilon$ -neighborhood of Γ/ε on the boundary $\partial \mathfrak{S}$.

First, we solve, given a small ϕ , problem (5.5) and (5.7) for ψ . Assume now that ϕ satisfies the following decay property

$$|\nabla \phi(y)| + |\phi(y)| \leq e^{-\gamma/\varepsilon} \quad \text{if } |s| > \sigma/\varepsilon, \quad (5.10)$$

for certain constant $\gamma > 0$. The solvability can be done in the following way: let us observe that W is exponentially small for $|s| > \sigma/\varepsilon$, where s is the normal coordinate to Γ/ε . Then the problem

$$\Delta \psi - [1 - (1 - \eta_\delta^\varepsilon)pW^{p-1}]\psi = h \quad \text{in } \Omega_\varepsilon,$$

$$\frac{\partial \psi}{\partial \mathbf{n}_\varepsilon} = -(1 - \eta_\delta^\varepsilon) \frac{\partial W}{\partial \mathbf{n}_\varepsilon} - \varepsilon \frac{\partial \eta_{3\delta}^\varepsilon}{\partial \mathbf{n}_\varepsilon} \phi \quad \text{on } \partial \Omega_\varepsilon,$$

has a unique bounded solution ψ whenever $\|h\|_\infty \leq +\infty$. Moreover,

$$\|\psi\|_\infty \leq C\|h\|_\infty.$$

Since \tilde{N} is power-like with power greater than one, a direct application of contraction mapping principle yields that (5.5) and (5.7) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\|\psi(\phi)\|_{L^\infty} \leq C\varepsilon \left[\|\phi\|_{L^\infty(|s|>\sigma/\varepsilon)} + \|\nabla \phi\|_{L^\infty(|s|>\sigma/\varepsilon)} + e^{-\sigma/\varepsilon} \right], \quad (5.11)$$

where $|s| > \sigma/\varepsilon$ denotes the complement in Ω_ε of σ/ε -neighborhood of Γ/ε . Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty} \leq C\varepsilon \left[\|\phi_1 - \phi_2\|_{L^\infty(|s| > \sigma/\varepsilon)} + \|\nabla\phi_1 - \nabla\phi_2\|_{L^\infty(|s| > \sigma/\varepsilon)} \right]. \quad (5.12)$$

Therefore, from the above discussion, the full problem has been reduced to solving the following (nonlocal) problem in the infinite strip \mathfrak{S}

$$\mathcal{L}_2(\phi) = \eta_\delta^\varepsilon \left[-\tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi(\phi)) - \tilde{E} - pW^{p-1}\psi(\phi) \right] \quad \text{in } \mathfrak{S}, \quad (5.13)$$

$$\mathfrak{B}(\phi) + \eta_\delta^\varepsilon \frac{\partial W}{\partial \mathbf{n}_\varepsilon} = 0 \quad \text{on } \partial\mathfrak{S}, \quad (5.14)$$

for $\phi \in W^{2,q}(\mathfrak{S})$ satisfying condition (5.10). Here \mathcal{L}_2 denotes a linear operator that coincides with $\tilde{\mathcal{L}}$ on the region $|s| < 8\sigma/\varepsilon$, \mathfrak{B} denotes the inward normal derivatives of \mathfrak{S} that coincides with inward normal $\partial/\partial \mathbf{n}_\varepsilon$ of Ω_ε on the region $|s| < 8\sigma/\varepsilon$.

The definitions of these operators can be shown as follows. The operator $\tilde{\mathcal{L}}$ for $|s| < 8\sigma/\varepsilon$ is given in coordinates (x, z) by formula (3.33). We extend it for functions ϕ defined in the strip \mathfrak{S} in terms of (x, z) as the following

$$\mathcal{L}_2(\phi) = L_2(\phi) + \chi(\varepsilon|x|)B(\phi) \quad \text{in } \mathfrak{S} \quad (5.15)$$

where $\chi(r)$ is a smooth cut-off function which equals 1 for $0 \leq r < 10\sigma$ and vanishes identically for $r > 20\sigma$, L_2 and B are the operators defined in (4.30) and (3.34). Similarly, the boundary conditions can be written as

$$\chi(\varepsilon|x|)\tilde{D}_0(\phi) - \frac{\partial\phi}{\partial\tau_\varepsilon} + \chi(\varepsilon|x|)\tilde{D}(W + \phi) = \chi(\varepsilon|x|)g \quad \text{on } \partial\mathfrak{S} \quad (5.16)$$

where the operators \tilde{D}_0 and \tilde{D} are defined in (4.31) and (4.8), τ_ε is the unit inward normal of $\partial\mathfrak{S}$.

Rather than solving problem (5.13)-(5.14) satisfying the boundary condition, we deal with the following projected problem: for each pair of parameters f_2 and e in F , finding functions $\phi \in W^{2,q}(\mathfrak{S})$, $c, d \in L^q(\Gamma)$ and Λ_1, Λ_2 such that

$$\mathcal{L}_2(\phi) = -\chi E_1 - \chi\tilde{N}_3(\phi) + c(\varepsilon z)\chi w_x + d(\varepsilon z)\chi Z \quad \text{in } \mathfrak{S}, \quad (5.17)$$

$$\chi\tilde{D}_0(\phi) - \frac{\partial\phi}{\partial\tau_\varepsilon} + \chi\tilde{D}(W + \phi) = \chi g \quad \text{on } \partial\mathfrak{S}, \quad (5.18)$$

$$\int_{\mathbb{R}} \phi(x, z) w_x dx = \Lambda_1, \quad \int_{\mathbb{R}} \phi(x, z) Z(x) dx = \Lambda_2 \quad \text{in } \Gamma_\varepsilon, \quad (5.19)$$

where $\tilde{N}_3(\phi) = \tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi(\phi)) + pW^{p-1}\psi(\phi)$. In Sections 6 and 7, we will prove that this problem has a unique solution ϕ whose norm is controlled by the $\|\cdot\|_{q,\varrho}$ -norm, not of the error component

E_{11} , but rather of the components E_{12} and g . Moreover, ϕ will satisfies (5.10). The reader can refer to the conclusion in Proposition 7.1.

After this has been done, our task is to adjust the parameters f_2 and e such that the functions c and d are identically zero. It is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the pair (f_2, e) with suitable boundary conditions. Indeed, we will derive the system of differential equations for the unknown functions f_2, e in Section 8, and then show the solvability of this system on the infinite dimensional space F (cf. (3.38)) in Section 9.

6 The Invertibility of \mathcal{L}_2

Let \mathcal{L}_2 be the operator defined by (5.15) and g be the functions in (4.34). Note that the function $\chi(\varepsilon|x|)$ is even in the definition of \mathcal{L}_2 . In this section, We study the following linear problem: for given $h \in L^q(\mathfrak{S})$ and $g \in L^q(\partial\mathfrak{S})$, finding functions $\phi \in W^{2,q}(\mathfrak{S})$, $c, d \in L^q(\partial\mathfrak{S})$ and Λ_1, Λ_2 such that

$$\begin{aligned} \mathcal{L}_2(\phi) &= \chi h + c(\varepsilon z) \chi w_x + d(\varepsilon z) \chi Z \quad \text{in } \mathfrak{S}, \\ \chi \tilde{D}_0(\phi) - \frac{\partial \phi}{\partial \tau_\varepsilon} + \chi \tilde{D}(W + \phi) &= \chi g \quad \text{on } \partial\mathfrak{S}, \\ \int_{\mathbb{R}} \phi w_x(x) dx &= \Lambda_1, \quad \int_{\mathbb{R}} \phi Z(x) dx = \Lambda_2 \quad \text{in } \Gamma_\varepsilon. \end{aligned} \tag{6.1}$$

Proposition 6.1. *If σ in the definition of \mathcal{L}_2 is chosen small enough and $h \in L^q(\mathfrak{S})$, then there exists a constant $C > 0$, independent of ε , such that for all small ε , the problem (6.1) has a unique solution $\phi = T_1(h, g)$ with suitable Λ_1 and Λ_2 which satisfy*

$$\begin{aligned} \|\phi\|_{q, \varrho} &\leq C(\|h\|_{L^q(\mathfrak{S})} + \|g\|_{L^q(\partial\mathfrak{S})}), \\ \|\Lambda_i\|_{L^q(\Gamma_\varepsilon)} &\leq C(\|h\|_{L^q(\mathfrak{S})} + \|g\|_{L^q(\partial\mathfrak{S})}), \quad \forall i = 1, 2. \end{aligned} \tag{6.2}$$

Moreover, if h, g have compact supports contained in $|x| \leq 20\sigma/\varepsilon$, then

$$|\phi(x, z)| + |\nabla \phi(x, z)| \leq \|\phi\|_{L^\infty} e^{-2\sigma/\varepsilon} \quad \text{for } |x| > 40\sigma/\varepsilon. \tag{6.3}$$

Proof. Note that the problem can be written as

$$\begin{aligned} \phi_{xx} + \Delta^{\Gamma_\varepsilon} \phi - \phi + p w^{p-1} \phi &= -p(W^{p-1} - w^{p-1})\phi - \chi B(\phi) \\ &\quad + \chi h + c(\varepsilon z) \chi w_x + d(\varepsilon z) \chi Z \quad \text{in } \mathfrak{S}, \\ \frac{\partial \phi}{\partial \tau_\varepsilon} &= -\chi g + \chi \tilde{D}_0(\phi) + \chi \tilde{D}(W + \phi) \quad \text{on } \partial\mathfrak{S}, \\ \int_{\mathbb{R}} \phi w_x dx &= \Lambda_1, \quad \int_{\mathbb{R}} \phi Z(x) dx = \Lambda_2 \quad \text{in } \Gamma_\varepsilon. \end{aligned}$$

Let

$$\varphi = T_1 \left(\chi h - p(W^{p-1} - w^{p-1})\phi - \chi B(\phi), G \right),$$

where

$$G(\phi) = -\chi g + \chi \tilde{D}_0(\phi) + \chi \tilde{D}(W + \phi),$$

and T_1 is the bounded operator defined by Lemma B.2.4.

The key point is that the operator

$$\tilde{B}_4(\phi) = -\chi B(\phi) - p(W^{p-1} - w^{p-1})\phi,$$

is small in the sense that

$$\|\tilde{B}_4(\phi)\|_{q,\varrho} \leq C\sigma\|\phi\|_{q,\varrho}.$$

Similar results hold for $G(\phi)$. Hence, the results can be derived by the invertibility conclusion of Lemma B.2.4 if we choose σ sufficiently small.

Since χ is supported on $|x| < 20\sigma/\varepsilon$, then ϕ satisfies for $|x| > 40\sigma/\varepsilon$ a problem of the form

$$\phi_{xx} + \Delta^{\Gamma_\varepsilon} \phi - (1 + o(1))\phi = 0 \quad \text{for } |x| > 40\sigma/\varepsilon, \quad \frac{\partial \phi}{\partial \tau_\varepsilon} = 0.$$

Hence, the validity of formula (6.3) can be showed easily. \square

7 Solving the Nonlinear Projection Problem

In this section, we will solve (5.17)-(5.19) in \mathfrak{S} . A first elementary, but crucial observation is the following: The term

$$E_{11} = \varepsilon^3 \Delta^\Gamma eZ + \varepsilon \lambda_0 eZ,$$

in the decomposition of E_1 , has precisely the form $d(\varepsilon z)Z$ and can be absorbed in that term $\chi d(\varepsilon z)Z$. Then, the equivalent equation of (5.17) is

$$\mathcal{L}_2(\phi) = \chi E_{12} + \chi \tilde{N}_3(\phi) + c(\varepsilon z) \chi w_x + d(\varepsilon z) \chi Z.$$

Let T_1 be the bounded operator defined by Proposition 6.1. Then the problem (5.17)-(5.19) is equivalent to the following fixed point problem

$$\phi = T_1(\chi E_{12} + \chi \tilde{N}_3(\phi), \chi g) \equiv \mathcal{A}(\phi). \quad (7.4)$$

We collect some useful facts to find the domain of the operator \mathcal{A} such that \mathcal{A} becomes a contraction mapping.

The big difference between E_{11} and E_{12} is their sizes. From (4.37) and (4.38)

$$\|E_{12}\|_{q,\varrho} \leq c_* \varepsilon^{\frac{1}{2}+4-\frac{3}{q}}, \quad (7.5)$$

while E_{11} is only of size $O(\varepsilon^{1+1/2-3/q})$. Similarly, we have

$$\|g\|_{q,\varrho;\partial\mathfrak{S}} \leq c_* \varepsilon^{\frac{1}{2}+4-\frac{3}{q}}. \quad (7.6)$$

The operator T_1 has a useful property: assume \hat{h} has a support contained in $|x| \leq 20\sigma/\varepsilon$, then $\phi = T_1(\hat{h})$ satisfies the estimate

$$|\phi| + |\nabla\phi| \leq \|\phi\|_{\infty,\varrho} e^{-2\sigma/\varepsilon} \quad \text{for } |x| > 40\sigma/\varepsilon. \quad (7.7)$$

Recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition

$$\|\psi(\phi)\|_{\infty,\varrho} \leq C\varepsilon \left[\left\| |\phi| + |\nabla\phi| \right\|_{L^\infty(|x|>20\sigma/\varepsilon)} + e^{-\sigma/\varepsilon} \right],$$

and a Lipschitz condition of the form

$$\|\psi(\phi) - \psi(\tilde{\phi})\|_{\infty,\varrho} \leq C\varepsilon \left[\left\| |\phi - \tilde{\phi}| + |\nabla(\phi - \tilde{\phi})| \right\|_{L^\infty(|x|>20\sigma/\varepsilon)} \right].$$

Now, the facts above will allow us to construct a region where contraction mapping principle applies and then solve the problem (5.17)-(5.19). Consider the following closed, bounded subset

$$\mathfrak{D} = \left\{ \phi \in H^2(\mathfrak{S}) \left| \begin{array}{l} \|\phi\|_{2,q,\varrho} \leq \varsigma \varepsilon^{\frac{1}{2}+4-\frac{3}{q}}, \\ \left\| |\phi| + |\nabla\phi| \right\|_{L^\infty(|x|>40\sigma/\varepsilon)} \leq \|\phi\|_{2,q,\varrho} e^{-\sigma/\varepsilon}. \end{array} \right. \right\} \quad (7.8)$$

We claim that if the constant ς is sufficiently large, then the map \mathcal{A} defined in (7.4) is a contraction from \mathfrak{D} into itself. Let us analyze the Lipschitz character of the nonlinear operator involved in \mathcal{A} for functions in \mathfrak{D}

$$\begin{aligned} \chi \tilde{N}_3(\phi) &= \chi \tilde{N}_1(\phi + \psi(\phi)) + \chi p W^{p-1} \psi(\phi) \\ &\equiv \bar{N}_3(\phi) + \chi p W^{p-1} \psi(\phi). \end{aligned}$$

Note that $\tilde{N}_1(\varphi) = p[(W + t\varphi)^{p-1} - W^{p-1}]\varphi$ for some $t \in (0, 1)$. From here it follows that

$$|\tilde{N}_1(\varphi)| \leq C(|\varphi|^p + |\varphi|^2).$$

Denoting $S_\sigma = \mathfrak{S} \cap \{|x| < 10\sigma/\varepsilon\}$, we have that for $\phi \in \mathfrak{D}$

$$\|\bar{N}_3(\phi)\|_{q,\varrho} \leq C \left[\|\phi\|_{qp,\varrho}^p + \|\phi\|_{2q,\varrho}^2 + \|\psi(\phi)\|_{qp,\varrho;S_\sigma}^p + \|\psi(\phi)\|_{2q,\varrho;S_\sigma}^2 \right].$$

Using Sobolev's embedding, we derive

$$\|\phi\|_{qp,\varrho}^p + \|\phi\|_{2q,\varrho}^2 \leq C \left(\|\phi\|_{2,q,\varrho}^p + \|\phi\|_{2,q,\varrho}^2 \right).$$

Using estimates (5.11), the facts that $\phi \in \mathfrak{D}$, (7.7), that of the area of S_σ is of order $O(\sigma/\varepsilon)$ and Sobolev's embedding, we get

$$\|\psi(\phi)\|_{qp,\varepsilon;S_\sigma}^p + \|\psi(\phi)\|_{2q,\varepsilon;S_\sigma}^2 \leq C e^{-\sigma/4\varepsilon} \left[1 + \|\phi\|_{2,q,\varepsilon}^p + \|\phi\|_{2,q,\varepsilon}^2 \right].$$

Hence, from the properties of W and $\psi(\phi)$ we obtain

$$\|\chi \tilde{N}_3(\phi)\|_{qp,\varepsilon} \leq C(\varepsilon^{(\frac{1}{2}+4-\frac{3}{q})p} \zeta^p + \varepsilon^{9-\frac{6}{q}} \zeta^2). \quad (7.9)$$

As for Lipschitz condition, after a direct calculation we find

$$\begin{aligned} \|\tilde{N}_1(\varphi_1) - \tilde{N}_1(\varphi_2)\|_{q,\varepsilon} &\leq C \left[\|\varphi_1\|_{qp,\varepsilon}^{p-1} + \|\varphi_1\|_{2q,\varepsilon} + \|\varphi_2\|_{qp,\varepsilon}^{p-1} + \|\varphi_2\|_{2q,\varepsilon} \right] \\ &\quad \times (\|\varphi_1 - \varphi_2\|_{qp,\varepsilon} + \|\varphi_1 - \varphi_2\|_{2q,\varepsilon}). \end{aligned}$$

Hence,

$$\begin{aligned} \|\bar{N}_3(\phi) - \bar{N}_3(\tilde{\phi})\|_{q,\varepsilon} &\leq \|N_1(\phi + \psi(\phi)) - N_1(\tilde{\phi} + \psi(\phi))\|_{q,\varepsilon;S_\sigma} \\ &\quad + \|N_1(\tilde{\phi} + \psi(\phi)) - N_1(\tilde{\phi} + \psi(\tilde{\phi}))\|_{q,\varepsilon;S_\sigma} \\ &\leq v \left(\|\phi - \tilde{\phi}\|_{2q,\varepsilon;S_\sigma} + \|\phi - \tilde{\phi}\|_{qp,\varepsilon;S_\sigma} \right) \\ &\quad + v \left(\|\psi(\phi) - \psi(\tilde{\phi})\|_{2q,\varepsilon;S_\sigma} + \|\psi(\phi) - \psi(\tilde{\phi})\|_{qp,\varepsilon;S_\sigma} \right), \end{aligned}$$

where $v = v_1 + v_2$ with

$$v_l = \|\phi_l\|_{qp,\varepsilon;S_\sigma}^{p-1} + \|\psi(\phi_l)\|_{qp,\varepsilon;S_\sigma}^{p-1} + \|\phi_l\|_{2q,\varepsilon;S_\sigma} + \|\psi(\phi_l)\|_{2q,\varepsilon;S_\sigma}.$$

Arguing as above and using the Lipschitz dependence of ψ on ϕ , it can be derived

$$\|\chi \tilde{N}_3(\phi) - \chi \tilde{N}_3(\tilde{\phi})\|_{q,\varepsilon} \leq C(\varepsilon^{(\frac{1}{2}+4-\frac{3}{q})(p-1)} \zeta^{p-1} + \varepsilon^{\frac{1}{2}+4-\frac{3}{q}} \zeta) \|\phi - \tilde{\phi}\|_{2,q,\varepsilon}. \quad (7.10)$$

Now, we can find the solution of (7.4) in the sequel. Let $\phi \in \mathfrak{D}$ and $\nu = \mathcal{A}(\phi)$, then from (7.5)-(7.6) and (7.9)

$$\|\nu\|_{2,q,\varepsilon} \leq \|T_1\| \left[c_* \varepsilon^{\frac{1}{2}+4-\frac{3}{q}} + C \zeta^p \varepsilon^{(\frac{1}{2}+4-\frac{3}{q})p} + C \zeta^2 \varepsilon^{9-\frac{6}{q}} \right].$$

Choosing any number $\varsigma > C_* \|T_1\|$, we get that for small ε

$$\|\nu\|_{2,q,\varepsilon} \leq \varsigma \varepsilon^{\frac{1}{2}+4-\frac{3}{q}}.$$

From (7.7)

$$\left\| |\nu| + |\nabla \nu| \right\|_{L^\infty(|x|>40\sigma/\varepsilon)} \leq \|\nu\|_\infty e^{-2\sigma/\varepsilon} \leq \|\nu\|_{2,q,\varepsilon} e^{-\sigma/\varepsilon}.$$

Therefore, $\nu \in \mathfrak{D}$. \mathcal{A} is clearly a contraction thanks to (7.10) and we can conclude that (7.4) has a unique solution in \mathfrak{D} .

The error E_{12} and the operator T_1 itself carry the functions f_2 and e as parameters. For future reference, we should consider their Lipschitz dependence on these parameters. (4.40) is just the formula about the Lipschitz dependence of error E_{12} on these two parameters. The other task can be realized by careful and direct computations of all terms involved in the differential operator which will show this dependence is indeed Lipschitz with respect to the $W^{2,q}$ -norm (for all ε).

Within the operator, consider for instance the following term involving $\Delta^\Gamma f_2$

$$Q_{f_2}(\phi) = \varepsilon^4 \Delta^\Gamma f_2 \phi_x.$$

Then we have

$$\|Q_{f_2}(\phi)\|_{L^q(B)}^q \leq \varepsilon^{4q-2} \int_\Gamma |\Delta^\Gamma f_2|^q \left(\sup_z \int_{\mathbb{R}} |\phi_x(x, z)|^q dx \right).$$

Let $\mu(z) = \int_{\mathbb{R}} |\phi_x(x, z)|^q dx$. Then there holds

$$\begin{aligned} \sup_z \mu(z) &\leq \varepsilon \int_{\mathfrak{S}} |\phi_x|^q dx + \frac{1}{\varepsilon} \int_{\mathfrak{S}} |\phi_x|^{q-1} |\nabla^\Gamma \phi_x| dx \\ &\leq \frac{1}{2} \sup_z \mu(z) + \frac{C}{\varepsilon^2} \int_{\mathfrak{S}} |\nabla^\Gamma \phi_x|^q dx, \end{aligned} \tag{7.11}$$

and we can obtain

$$\mu(z) \leq C\varepsilon^{-2} \|\phi\|_{2,q,\varrho}^q.$$

Therefore,

$$\|Q_{f_2}(\phi)\|_{q,\varrho} \leq C\varepsilon \|f_2\|_a.$$

Similar estimates can be applied to other terms in the operator involving $\Delta^\Gamma f_2$.

For the linear operator T_1 , we have the following Lipschitz dependence

$$\|T_1(f_2) - T_1(\tilde{f}_2)\|_{2,q,\varrho} \leq C\varepsilon \|f_2 - \tilde{f}_2\|_a.$$

Moreover, the operator \tilde{N}_3 also has Lipschitz dependence on (f_2, e) . It is easily checked that for $\phi \in \mathfrak{D}$ we have, with obvious notation

$$\|\chi \tilde{N}_{3,(f_2,e)}(\phi) - \chi \tilde{N}_{3,(\tilde{f}_2,\tilde{e})}(\phi)\|_{q,\varrho} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}} \left[\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b \right].$$

Hence, from the fixed point characterization we get that

$$\|\phi(f_2, e) - \phi(\tilde{f}_2, \tilde{e})\|_{2,q,\varrho} \leq C\varepsilon^{\frac{1}{2}+4-\frac{3}{q}} \left[\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b \right]. \tag{7.12}$$

As a conclusion, we give the proposition.

Proposition 7.1. *There is a number $\varsigma > 0$ such that for all ε small enough and all parameters (f_2, e) in F , problem (5.17)-(5.19) has a unique solution $\phi = \phi(f_2, e)$ which satisfies*

$$\|\phi\|_{2,q,\varrho} \leq \varsigma \varepsilon^{\frac{1}{2}+4-\frac{3}{q}}, \quad \|\Lambda_i\|_{q,\varrho;\Gamma_\varepsilon} \leq \varsigma \varepsilon^{\frac{1}{2}+4-\frac{3}{q}}, \quad \forall i = 1, 2,$$

$$\left\| |\phi| + |\nabla\phi| \right\|_{L^\infty(|x|>40\sigma/\varepsilon)} \leq \|\phi\|_{2,q,\varrho} e^{-\sigma/\varepsilon}.$$

Moreover, $\phi(f_2, e)$ depends Lipschitz-continuously on the parameters f_2 and e in the sense of the estimate (7.12).

□

8 Estimates of the Projections Against w_x and Z

As we mentioned in Section 5, we will set up the system of differential equations for the unknown parameters f_2 and e defined on Γ which are equivalent to making c, d zero in the system (5.17)-(5.19). On the boundary, we have imposed the conditions (4.22) and (4.18) for the parameters f_2 and e , which are restated respectively in the following

$$\nabla_\rho f_2 + I(\varepsilon\theta)f_2 + \Re(\theta) = 0, \quad \nabla_\rho e + \frac{1}{2}I(\varepsilon\theta)e = 0. \quad (8.1)$$

On the interior of Γ , these equations are obtained by simply integrating the equations (5.17) (only in x) against w_x and Z respectively. It is easy to derive the following equations

$$\int_{\mathbb{R}} \left[\chi E_1 + \chi \tilde{N}_3(\phi) + \Delta^{\Gamma_\varepsilon} \phi + \chi B(\phi) + p(W^{p-1} - w^{p-1})\phi \right] w_x dx = 0, \quad (8.2)$$

$$\int_{\mathbb{R}} \left[\chi E_1 + \chi \tilde{N}_3(\phi) + \Delta^{\Gamma_\varepsilon} \phi + \lambda_0 \phi + \chi B(\phi) + p(W^{p-1} - w^{p-1})\phi \right] Z dx = 0, \quad (8.3)$$

where the error term E_1 is defined in (4.32), the operators \tilde{N}_3 and B are defined in (5.17) and (3.34). It is crucial to estimate the terms

$$\int_{\mathbb{R}} E_1 w_x dx \quad \text{and} \quad \int_{\mathbb{R}} E_1 Z dx.$$

The same arguments can be applied to other terms in (8.2) and (8.3). Now, we divide the estimates for the components in (8.2) and (8.3) into several parts.

8.1 Estimates of the projection against w_x

First, multiplying (4.32) by w_x and integrating over the variable x , using the decomposition of E_1 in (4.35) and the fact that w_x is an odd function in x , we obtain

$$\int_{\mathbb{R}} E_1 w_x dx = \int_{\mathbb{R}} E_{12} w_x dx.$$

More precisely, there holds

$$\begin{aligned} \int_{\mathbb{R}} E_{12} w_x dx &= \varepsilon^4 \int_{\mathbb{R}} [\check{\mathbf{B}}_4 + \check{G}_4 + \check{\mathfrak{D}}_4 + p(p-1)w^{p-2}eZ\phi_3] w_x dx \\ &+ \int_{\mathbb{R}} \left[\sum_{i=4}^6 \varepsilon^i S_i + \sum_{i=4}^6 \varepsilon^i T_i + \sum_{i=5}^{10} \varepsilon^i G_i + \varepsilon^5 \mathbf{B}_5 \right] w_x dx. \end{aligned}$$

By direct calculation, we derive that

$$\begin{aligned} \int_{\mathbb{R}} E_{12} w_x dx &= -\varepsilon^4 \sigma_1 \left[\Delta^\Gamma f_2 + (k_1^2 + k_2^2) f_2 \right] + \gamma_1 \varepsilon^4 e + \gamma_2 \varepsilon^6 \Delta^\Gamma e \\ &\quad + \varepsilon^5 b_{1\varepsilon} \Delta^\Gamma e + \varepsilon^5 b_{2\varepsilon} \Delta^\Gamma f_2 + \sum_{i=5}^{10} \varepsilon^i b_{i\varepsilon}, \end{aligned} \quad (8.4)$$

where γ_1, γ_2 are two constants and

$$\sigma_1 = \int_{\mathbb{R}} w_x^2 dx.$$

Here and below we denote by $b_{l\varepsilon}, l = 1, 2, i$, generic, uniformly bounded continuous functions of the form

$$b_{l\varepsilon} = b_{l\varepsilon}(f_2, e, \nabla^\Gamma f_2, \nabla^\Gamma e),$$

where additionally $b_{l\varepsilon}$ is uniformly Lipschitz in its arguments.

8.2 Projection of terms involving ϕ

We will estimate other terms that involve ϕ in (8.2),

$$\int_{\mathbb{R}} \left[\chi N_2(\phi) + \Delta^{\Gamma_\varepsilon} \phi + \chi B(\phi) + p(W^{p-1} - w^{p-1}) \phi \right] w_x dx. \quad (8.5)$$

Using the condition in (5.19), we first estimate $\int_{\mathbb{R}} \Delta^{\Gamma_\varepsilon} \phi w_x dx$, the estimate for the term can be done as follows

$$\Upsilon_1(z) = \int_{\mathbb{R}} \Delta^{\Gamma_\varepsilon} \phi w_x dx = \Delta^{\Gamma_\varepsilon} \int_{\mathbb{R}} \phi w_x dx \equiv \Delta^{\Gamma_\varepsilon} \Lambda_1.$$

With the help of the Proposition 7.1, we have the following estimate

$$\|\Upsilon_1\|_{L^q(\Gamma_\varepsilon)} \leq C \varepsilon^{\frac{1}{2} + 4 - \frac{3}{q}}.$$

The last two components in (8.5) are

$$\Upsilon_2 = \int_{\mathbb{R}} B(\phi) w_x dx \quad \text{and} \quad \Upsilon_3 = \int_{\mathbb{R}} p(W^{p-1} - w^{p-1}) \phi w_x dx.$$

Here we recalled the definitions of the operator B in (3.34) and the local approximation W in (5.1). We make the following observation: all terms in $B(\phi)$ carry ε^2 and involve powers of x times derivatives of 1, 2 or two orders of ϕ . The conclusion is that since w_x has exponential decay then

$$\int_{\Gamma} |\Upsilon_2(\theta)|^q d\theta \leq C \varepsilon^4 \|\phi\|_{2,q,\varrho}^q.$$

Hence there holds

$$\|\Upsilon_2\|_{L^q(\Gamma)} \leq C \varepsilon^{\frac{1}{2} + 4 + \frac{1}{q}}.$$

In $B(\phi)$ we single out two less regular terms. The one whose coefficient depends on $\Delta^\Gamma f_2$ explicitly has the form

$$\begin{aligned}\Upsilon_{2*} &= \varepsilon^k \Delta^\Gamma f_2 \int_{\mathbb{R}} \phi_x Z \left(1 + \varepsilon \left(x - \sum_{l=0}^2 \varepsilon^l f_l \right)^{-2} \right) \\ &= -\varepsilon^k \Delta^\Gamma f_2 \int_{\mathbb{R}} \phi \left\{ Z \left(1 + \varepsilon \left(x - \sum_{l=0}^2 \varepsilon^l f_l \right)^{-2} \right) \right\}_x.\end{aligned}$$

Since ϕ has Lipschitz dependence on (f_2, e) in the form (7.12), we see that

$$\|\Upsilon_{2*}(f_2, e) - \Upsilon_{2*}(\tilde{f}_2, \tilde{e})\|_{L^q(\Gamma)} \leq C \varepsilon^{\frac{1}{2}+4-\frac{3}{q}} (\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b). \quad (8.6)$$

The other arising from second derivative in y for ϕ is

$$\Upsilon_{2**} = \int_{\mathbb{R}} \Delta^\Gamma \phi w_x \left[1 - \left(1 + \varepsilon \left(x - \sum_{l=0}^2 \varepsilon^l f_l \right)^{-2} \right) \right] dx.$$

We readily see that

$$\|\Upsilon_{2**}(f_2, e) - \Upsilon_{2**}(\tilde{f}_2, \tilde{e})\|_{L^q(\Gamma)} \leq C \varepsilon^{\frac{1}{2}+4+\frac{1}{q}} (\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b). \quad (8.7)$$

The remainder $\Upsilon_2 - \Upsilon_{2*} - \Upsilon_{2**}$ actually defines for fixed ε a compact operator of the pair (f_2, e) into $L^q(\Gamma)$. This is a consequence of the fact that weak convergence in $W^{2,q}(\mathfrak{S})$ implies local strong convergence in $W^{1,q}(\mathfrak{S})$. If $f_{2,j}$ and e_j are weakly convergent sequences in $W^{2,q}(\mathfrak{S})$ then clearly the functions $\phi(f_{2,j}, e_j)$ constitute a bounded sequence in $W^{1,q}(\mathfrak{S})$. In the above remainder one can integrate by parts if necessary once in x . Averaging against w_x which decays exponentially localizes the situation and the desired result follows.

Let us consider now the term

$$\Upsilon_3(z) = \int_{\mathbb{R}} p(W^{p-1} - w^{p-1}) \phi w_x dx.$$

Since the term $W = w(x) + \varepsilon e Z + \sum_{i=2}^3 \varepsilon^i \varphi_i + \sum_{i=1}^3 \phi_i$ can be estimated as

$$\varepsilon |e(\varepsilon z) Z(x)| + \sum_{i=2}^3 |\varepsilon^i \varphi_i| + \sum_{i=1}^3 |\phi_i| \leq C \varepsilon (1 + |x|^2) e^{-|x|},$$

we easily see that for some $\kappa > 0$ the uniform bound holds

$$|W^{p-1} - w^{p-1}| \cdot |w_x| \leq C \varepsilon e^{-\kappa|x|}.$$

From here we readily find that

$$\|\Upsilon_3\|_{L^q(\Gamma)} \leq C \varepsilon^{\frac{4}{q}} \|\phi\|_{2,q,\varrho} \leq C \varepsilon^{\frac{1}{2}+4+\frac{1}{q}}.$$

We observe also that the term in (8.5) such as

$$\Upsilon_4(z) = \int_{\mathbb{R}} \chi \tilde{N}_3(\phi) w_x dx,$$

can be estimated similarly. In fact, using the definition of $\tilde{N}_3(\phi)$ and the exponential decay of w_x we obtain

$$\|\Upsilon_4\|_{L^q(\Gamma)} \leq C \|\phi\|_{2,q,\varrho}^2 \leq C \varepsilon^{9-\frac{6}{q}}.$$

These terms define compact operators similarly as before.

8.3 Estimates of the projection against Z

We observe that exactly the same estimates can be carried out in the terms obtained from integration against Z . So the remaining thing is to compute the term $\int_{\mathbb{R}} E_1 Z dx$.

Multiplying (4.32) by Z and integrating over the variable x and using the decomposition of E_1 in (4.35), we get

$$\int_{\mathbb{R}} E_1 Z dx = \int_{\mathbb{R}} E_{11} Z dx + \int_{\mathbb{R}} E_{12} Z dx,$$

where

$$\int_{\mathbb{R}} E_{11} Z dx = \varepsilon(\varepsilon^2 \Delta^\Gamma e + \lambda_0 e) \int_{\mathbb{R}} Z^2 dx = \varepsilon^3 \Delta^\Gamma e + \varepsilon \lambda_0 e.$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{R}} E_{12} Z dx &= \varepsilon^4 \int_{\mathbb{R}} [\hat{\mathbf{B}}_4 + \hat{G}_4 + \hat{\mathbf{D}}_4 + p(p-1)w^{p-2}eZ\phi_3] Z dx \\ &\quad + \int_{\mathbb{R}} \left[\sum_{i=4}^6 \varepsilon^i S_i + \sum_{i=4}^6 \varepsilon^i T_i + \sum_{i=5}^{10} \varepsilon^i G_i + \varepsilon^5 \mathbf{B}_5 \right] Z dx. \end{aligned}$$

The components in $\hat{\mathbf{B}}_4, \hat{G}_4$ and $\hat{\mathbf{D}}_4$ are even functions in the variable x and independent of the parameters f_2 . Here, the function ϕ_3 has the form

$$\phi_3(x, z) = \psi_{31}(x, \varepsilon z) + \psi_{32}(x, \varepsilon z),$$

The components in ψ_{31} and ψ_{32} are independent of the parameters f_2 . Moreover, ψ_{31} is an odd function in the variable x and ψ_{32} is an even function in the variable x . Therefore, adding up all terms together, we conclude that

$$\int_{\mathbb{R}} E_{12} Z dx = \varepsilon^5 b_{1\varepsilon}^1 \Delta^\Gamma e + \varepsilon^5 b_{1\varepsilon}^2 \Delta^\Gamma f_2 + \sum_{i=5}^{10} \varepsilon^i b_{i\varepsilon}(f_2, e, \nabla^\Gamma f_2, \nabla^\Gamma e).$$

Here we denote by $b_{1\varepsilon}^j, j = 1, 2$ and $b_{i\varepsilon}$, generic, uniformly bounded continuous functions, moreover, $b_{i\varepsilon}$ is uniformly Lipschitz in its arguments.

9 Solving the System for (f_2, e) : Proof of Theorem 1.1

9.1 Proof of Theorem 1.1

Using the estimates in previous section, we find the following nonlinear, nonlocal system of differential equations for the parameters (f_2, e) in the variable $y = \varepsilon z \in \Gamma$

$$\mathcal{L}_1^*(f_2) \equiv \Delta^\Gamma f_2 + (k_1^2 + k_2^2)f_2 = \gamma_1 e + \gamma_2 \varepsilon^2 \Delta^\Gamma e + M_{1\varepsilon} \quad \text{in } \Gamma, \quad (9.1)$$

$$\mathcal{L}_{2\varepsilon}^*(e) \equiv -\varepsilon^2 \Delta^\Gamma e - \lambda_0 e = M_{2\varepsilon} \quad \text{in } \Gamma, \quad (9.2)$$

with the boundary conditions

$$\frac{\partial f_2}{\partial \tau} + I f_2 + \mathfrak{R} = 0 \quad \text{on } \partial\Gamma, \quad (9.3)$$

$$\frac{\partial e}{\partial \tau} + \frac{1}{2} I e = 0 \quad \text{on } \partial\Gamma, \quad (9.4)$$

where γ_1 and γ_2 are two constants defined in (8.4), I and \mathfrak{R} are smooth functions defined in (4.3) and (4.23) and τ denotes the inward normal of $\partial\Gamma$. The operators $M_{1\varepsilon}$ and $M_{2\varepsilon}$ can be decomposed in the following form

$$M_{l\varepsilon}(f_2, e) = A_{l\varepsilon}(f_2, e) + K_{l\varepsilon}(f_2, e), \quad l = 1, 2$$

where $K_{l\varepsilon}$ is uniformly bounded in $L^q(\Gamma)$ for (f_2, e) in F and is also compact. The operator $A_{l\varepsilon}$ is Lipschitz in this region, see (8.6)-(8.7),

$$\|A_{l\varepsilon}(f_2, e) - A_{l\varepsilon}(\tilde{f}_2, \tilde{e})\|_{L^q(\Gamma)} \leq C\varepsilon^{\frac{1}{2} + \frac{1}{q}} [\|f_2 - \tilde{f}_2\|_a + \|e - \tilde{e}\|_b]. \quad (9.5)$$

Before solving (9.1)-(9.4), some basic facts about the invertibility of corresponding linear operators are in order. We first consider the following problem

$$\mathcal{L}_1^*(f_2) = \Delta^\Gamma f_2 + (k_1^2 + k_2^2)f_2 = h \quad \text{in } \Gamma, \quad \frac{\partial f_2}{\partial \tau} + I f_2 = 0 \quad \text{on } \partial\Gamma. \quad (9.6)$$

Lemma 9.1. *Under the non-degeneracy condition of Γ in (1.3), if $h \in L^q(\Gamma)$ then there is a constant ε_0 for each $0 < \varepsilon < \varepsilon_0$, the problem(9.6) has a unique solution $f_2 \in W^{2,q}(\Gamma)$ with the property*

$$\|f_2\|_{L^\infty(\Gamma)} + \|\nabla^\Gamma f_2\|_{L^\infty(\Gamma)} + \|\Delta^\Gamma f_2\|_{L^q(\Gamma)} \leq C\|h\|_{L^q(\Gamma)}.$$

Proof. Under the non-degeneracy condition of Γ in (1.3), the existence and the a priori estimates can be easily proved. \square

We then consider the following problem

$$\mathcal{L}_{2\varepsilon}^*(e) = -\varepsilon^2 \Delta^\Gamma e - \lambda_0 e = g \quad \text{in } \Gamma, \quad \frac{\partial e}{\partial \tau} + \frac{1}{2} I e = 0 \quad \text{on } \partial\Gamma. \quad (9.7)$$

Lemma 9.2. *If $g \in L^q(\Gamma)$ then there exists $\varepsilon_0 > 0$ such that for a sequence $(\varepsilon_l)_l$ with $0 < \varepsilon_l < \varepsilon_0$ such that problem (9.7) has a unique solution $e \in W^{2,q}(\Gamma)$ which satisfies*

$$\|e\|_b \leq C \varepsilon_l^{-2} \|g\|_{L^q(\Gamma)}.$$

Moreover, if $g \in W^{2,q}(\Gamma)$ then

$$\varepsilon_l^2 \|\Delta^\Gamma e\|_{L^q(\Gamma)} + \varepsilon_l \|\nabla^\Gamma e\|_{L^q(\Gamma)} + \|e\|_{L^\infty(\Gamma)} \leq C \|g\|_{W^{2,q}(\Gamma)}. \quad (9.8)$$

The proof will be given in subsection 9.2. \square

We finally consider the following system

$$\begin{aligned} \mathcal{L}^*(f_2, e) &\equiv (\mathcal{L}_1^*(f), \mathcal{L}_{2\varepsilon}^*(e)) = (h, g) \quad \text{in } \Gamma \\ \frac{\partial f_2}{\partial \tau} + I f_2 &= \Xi, \quad \frac{\partial e}{\partial \tau} + \frac{1}{2} I e = 0 \quad \text{on } \partial\Gamma, \end{aligned} \quad (9.9)$$

where Ξ is a smooth function.

Lemma 9.3. *Under the nondegeneracy condition (1.3), if $h, g \in L^q(\Gamma)$ then for the sequence of the parameter ε in Lemma 9.2, there is a unique solution (f_2, e) in $W^{2,q}(\Gamma)$ to problem (9.9) which satisfies*

$$\|f_2\|_a + \|e\|_b \leq C [\|h\|_{L^q(\Gamma)} + \varepsilon^{-1} \|g\|_{L^q(\Gamma)} + \|\Xi\|_{L^q(\Gamma)}].$$

Proof. Under the non-degeneracy condition (1.3), there exist \bar{f}_2 and e_0 satisfying

$$\Delta^\Gamma \bar{f}_2 = 0, \quad \frac{\partial \bar{f}_2}{\partial \tau} + I \bar{f}_2 = \Xi.$$

Setting $f_2 = \tilde{f}_2 + \bar{f}_0$ to the system (9.9), the final conclusion can be derived from Lemma 9.1 and Lemma 9.2. \square

Proof of Theorem 1.1: Let $(\tilde{f}, \tilde{e}) \in F$, where F is defined in (3.38), and define

$$\begin{aligned} (h(f_2, e), g(f_2, e)) &= (\varepsilon A_{1\varepsilon}(f_2, e) + \varepsilon K_{1\varepsilon}(\tilde{f}_2, \tilde{e}), \varepsilon^2 A_{2\varepsilon}(f_2, e) + \varepsilon^2 K_{2\varepsilon}(\tilde{f}_2, \tilde{e})), \\ \Xi &= \Re(\tilde{f}_2, \tilde{e}). \end{aligned}$$

From (9.5), $A_{1\varepsilon}$ and $A_{2\varepsilon}$ are contraction mappings of its arguments in F . By Banach Contraction Mapping theorem and Lemma 9.3, we can solve the nonlinear problem

$$\mathcal{L}^*(f, e) \equiv (\mathcal{L}_1^*(f_2), \mathcal{L}_{2\varepsilon}^*(e)) = (h, g),$$

with the boundary conditions defined in (9.9) on the region F . Hence, we can define a new operator \mathcal{Z} from F into F by $\mathcal{Z}(\tilde{f}_2, \tilde{e}) = (f_2, e)$. Finding a solution to the problem (9.1)-(9.4) is equivalent

to locating a fixed point of \mathcal{Z} . Schauder's fixed point theorem applies to finish the proof of its existence. Hence, by Proposition 7.1 and the arguments followed, we complete the existence part of Theorem 1.1. Other properties of u_ε in Theorem 1.1 can be showed easily. \square

9.2 Proof of Lemma 9.2

To prove Lemma 9.2, we follow the method introduced in [12], which relies only on elementary considerations on the variational characterization of the eigenvalues of the operator $\mathcal{L}_{2\varepsilon}^*$ and the Weyl's asymptotic formula in (2.20). We remark this approach is a slightly different from [24] and [25] where Kato's theorems were the main tools.

First, we consider the following eigenvalue problem

$$\mathcal{L}_{2\varepsilon}^*(v) = -\varepsilon^2 \Delta^\Gamma v - \lambda_0 v = \Lambda_\varepsilon v \quad \text{in } \Gamma, \quad \frac{\partial v}{\partial \tau} + \frac{1}{2} I v = 0 \quad \text{on } \partial\Gamma. \quad (9.10)$$

We denote its eigenvalues $\Lambda_{\varepsilon,j}$ in non-decreasing order and counting them with multiplicity. Here λ_0 is the unique positive eigenvalue to the eigenvalue problem (1.5), which implies the spectrum of $\mathcal{L}_{2\varepsilon}^*$ contains negative or zero eigenvalues. From the Courant-Fisher characterization we can write $\Lambda_{\varepsilon,j}$ in two different ways:

$$\Lambda_{\varepsilon,j} = \sup_{\Xi \in \Xi_{j-1}} \left[\inf_{v \perp \Xi, v \neq 0} \frac{\int_\Gamma v \mathcal{L}_{2\varepsilon}^* v}{\int_\Gamma v^2} \right], \quad (9.11)$$

$$\Lambda_{\varepsilon,j} = \inf_{\Xi \in \Xi_j} \left[\sup_{v \in \Xi, v \neq 0} \frac{\int_\Gamma v \mathcal{L}_{2\varepsilon}^* v}{\int_\Gamma v^2} \right]. \quad (9.12)$$

Here Ξ_j (resp. Ξ_{j-1}) represents the family of j dimensional (resp. $j-1$ dimensional) subspaces of $H^2(\Gamma)$ constituting of functions defined on Γ with boundary condition in (9.10), and the symbol \perp denotes orthogonality with respect to the L^2 scalar product. There holds the following result for the estimates of gap between two successive eigenvalues.

Lemma 9.4. *There exists a number $\varepsilon_0 > 0$ such that for all $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ and all $j \geq 1$ the following estimate holds.*

$$\Lambda_{\varepsilon_1,j} = \frac{\varepsilon_1^2}{\varepsilon_2^2} \Lambda_{\varepsilon_2,j} - \lambda_0 \left(1 - \frac{\varepsilon_1^2}{\varepsilon_2^2} \right). \quad (9.13)$$

In particular, the functions $\varepsilon \in (0, \varepsilon_0) \mapsto \Lambda_{\varepsilon,j}$ are continuous and increasing.

Proof. Let us consider small numbers $0 < \varepsilon_1 < \varepsilon_2$. We observe that for any v with $\int_\Gamma v^2 = 1$, we have

$$\varepsilon_2^{-2} \int_\Gamma v \mathcal{L}_{2\varepsilon_2}^* v - \varepsilon_1^{-2} \int_\Gamma v \mathcal{L}_{2\varepsilon_1}^* v = \lambda_0 (\varepsilon_1^{-2} - \varepsilon_2^{-2}).$$

Then the result follows. \square

Proof of Lemma 9.2: For $\ell \in \mathbb{N}$, choose $\sigma_\ell = 2^{-\ell}$. In order to find a sequence of values $\varepsilon_\ell \in (\sigma_{\ell+1}, \sigma_\ell)$ such that the spectrums of the operators $\mathcal{L}_{2\varepsilon_\ell}^*$, for large ℓ , stay away from 0, we define

$$\mathcal{K}_\ell^1 = \{\varepsilon \in (\sigma_{\ell+1}, \sigma_\ell) : \ker \mathcal{L}_{2\varepsilon}^* \neq \emptyset\}, \quad \mathcal{K}_\ell^2 = (\sigma_{\ell+1}, \sigma_\ell) \setminus \mathcal{K}_\ell^1.$$

It is crucial to estimate the cardinality of \mathcal{K}_ℓ^1 . If $\varepsilon \in \mathcal{K}_\ell^1$ then for some j we have that $\Lambda_{\varepsilon,j} = 0$. The monotonicity of the function $\varepsilon \in (0, \varepsilon_0) \mapsto \Lambda_{\varepsilon,j}$ implies that $\Lambda_{\sigma_{\ell+1},j} < 0$. Hence,

$$\text{card}(\mathcal{K}_\ell^1) \leq N_{\sigma_{\ell+1}}, \quad (9.14)$$

where \mathbf{N}_ε is the number of negative eigenvalues of the operator $\mathcal{L}_{2\varepsilon}^*$.

We now give an asymptotic estimate on the number \mathbf{N}_ε of negative eigenvalues of the differential operator $\mathcal{L}_{2\varepsilon}^*$. By $(\rho_i)_i$ we will denote the set of eigenvalues of the eigenvalue problem

$$-\Delta^\Gamma \omega = \rho \omega \quad \text{in } \Gamma, \quad \frac{\partial \omega}{\partial \tau} + \frac{1}{2} I \omega = 0 \quad \text{on } \partial \Gamma.$$

From the Weyl asymptotic formula as those in (2.20) and the formula in (9.12), one derives

$$\mathbf{N}_\varepsilon \geq C_\Gamma (1 + o(1)) \varepsilon^{-2},$$

where C_Γ is a fixed constant depending on the volume of the manifold Γ and its dimension. To prove a similar upper bound, we choose i to be the first index such that $\varepsilon^2 \rho_i - \lambda_0 > 0$. Then from the Weyl formula we find that

$$i = C_\Gamma (1 + o(1)) \varepsilon^{-2}.$$

Define $\Xi_{j-1} = \text{span}\{\omega_\ell : \ell = 1, 2, \dots, j-1\}$. For an arbitrary function $v \in H^2(\Gamma)$ and $v \perp \Xi_{j-1}$, we can write

$$v = \sum_{\ell \geq j} \kappa_\ell \omega_\ell.$$

Plugging this v into (9.11) and using the Weyl formula, we also have

$$\mathbf{N}_\varepsilon \leq C_\Gamma (1 + o(1)) \varepsilon^{-2}.$$

Hence we get that

$$\mathbf{N}_\varepsilon \sim C_\Gamma \varepsilon^{-2} \quad \text{as } \varepsilon \rightarrow 0.$$

The last inequality and (9.14) imply that $\text{card}(\mathcal{K}_\ell^1) \leq C \sigma_\ell^{-2}$, and hence there exists an interval (a_ℓ, b_ℓ) such that

$$(a_\ell, b_\ell) \subset \mathcal{K}_\ell^2, \quad |b_\ell - a_\ell| \geq \frac{\text{meas}(\mathcal{K}_\ell^2)}{\text{card}(\mathcal{K}_\ell^1)} \geq 2C_0 \sigma_\ell^3, \quad (9.15)$$

for a universal positive constant C_0 , independent of ℓ . By setting $\varepsilon_\ell = (a_\ell + b_\ell)/2$ for all large $\ell \in \mathbb{N}$, we conclude that $\mathcal{L}_{2\varepsilon_\ell}^*$ is invertible and there exists a number $C > 0$, independent of ℓ , such that for all $j \in \mathbb{N}$ there holds

$$|\Lambda_{\varepsilon_\ell, j}| \geq C\varepsilon_\ell^2. \quad (9.16)$$

Assume the opposite, namely that for some j we have

$$|\Lambda_{\varepsilon_\ell, j}| < \delta\varepsilon_\ell^2,$$

with δ arbitrarily small. Since $\varepsilon_\ell \in \mathcal{K}_\ell^2$, then $|\Lambda_{\varepsilon_\ell, j}| > 0$. Let us assume that

$$0 < \Lambda_{\varepsilon_\ell, j} < \delta\varepsilon_\ell^2. \quad (9.17)$$

Then from Lemma 9.4, we have

$$\Lambda_{a_\ell, j} = \Lambda_{\varepsilon_\ell, j} - \frac{(\varepsilon_\ell^2 - a_\ell^2)}{\varepsilon_\ell^2}(\Lambda_{\varepsilon_\ell, j} + \lambda_0). \quad (9.18)$$

The inequalities in (9.15) and (9.17) imply that

$$\Lambda_{a_\ell, j} \leq \delta\varepsilon_\ell^2 - C_0\sigma_\ell^2 \frac{\sigma_\ell(\varepsilon_\ell + a_\ell)}{\varepsilon_\ell^2}(\Lambda_{\varepsilon_\ell, j} + \lambda_0) < 0,$$

if δ is chosen a priori sufficiently small. It follows from the continuity of the function $\varepsilon \in (0, \varepsilon_0) \mapsto \Lambda_{\varepsilon, j}$ that $\Lambda_{\varepsilon, j}$ must vanish at some $\varepsilon \in (a_\ell, \varepsilon_\ell)$, and we get a contradiction with the choice of the interval (a_ℓ, b_ℓ) .

The case

$$-\delta\sigma_\ell^2 < \Lambda_{\varepsilon_\ell, j} < 0$$

can be handled similarly. In fact, we have the inequality

$$\Lambda_{b_\ell, j} = \Lambda_{\varepsilon_\ell, j} + \frac{(b_\ell^2 - \varepsilon_\ell^2)}{\varepsilon_\ell^2}(\Lambda_{\varepsilon_\ell, j} + \lambda_0) > 0.$$

Hence, the proof of (9.16) for the spectral gap between critical eigenvalues was complete.

As a consequence, the solution to (9.2) exists and satisfies

$$\|e\|_{L^q(\Gamma)} \leq C\varepsilon_\ell^{-2} \|g\|_{L^q(\Gamma)}. \quad (9.19)$$

From (9.19) by a standard elliptic argument one can show

$$\varepsilon_\ell^2 \|\Delta^\Gamma e\|_{L^q(\Gamma)} + \varepsilon_\ell \|\nabla^\Gamma e\|_{L^q(\Gamma)} + \|e\|_{L^\infty(\Gamma)} \leq C\varepsilon_\ell^{-2} \|h_2\|_{L^q(\Gamma)}.$$

The reader can refer [12] for proof of further estimate in (9.8). \square

A A Linear Model Problem I

Recall that w is the even function defined in (1.4) and Z is the even eigenfunction defined in the eigenvalue problem (1.5). Recall that, Λ represents the strip in \mathbb{R}^3 with the notations

$$\begin{aligned}\Lambda &= \{ (x, \theta, \eta) : x \in \mathbb{R}, (\theta, \eta) \in \partial\mathbb{D} \times \mathbb{R}_+ \}, \\ \partial\Lambda &= \{ (x, \theta, \eta) : x \in \mathbb{R}, \theta \in \partial\mathbb{D}, \eta = 0 \}.\end{aligned}\tag{A.1}$$

We first consider the following linear problem

$$\phi_{xx}^0 + \frac{1}{l_1^2(\theta)}\phi_{\theta\theta}^0 + \frac{1}{l_2^2(\theta)}\phi_{\eta\eta}^0 - K\phi^0 + pw^{p-1}\phi^0 = 0 \quad \text{in } \Lambda,\tag{A.2}$$

$$\phi_\eta^0 = G(x, \theta) \quad \text{on } \partial\Lambda,\tag{A.3}$$

$$\int_{\mathbb{R}} \phi^0(x, \theta, \eta) w_x(x) dx = 0, \quad \int_{\mathbb{R}} \phi^0(x, \theta, \eta) Z(x) dx = 0 \quad \text{in } \Gamma_\varepsilon,\tag{A.4}$$

where $K > \lambda_0 + 1$ is a large positive constant, $l_1(\theta)$ and $l_2(\theta)$ are two smooth positive functions given in (4.3). Suppose the following orthogonality conditions hold

$$\int_{\mathbb{R}} G(x, \theta) w_x(x) dx = 0, \quad \int_{\mathbb{R}} G(x, \theta) Z(x) dx = 0.\tag{A.5}$$

Lemma A.1.1. *If $G \in L^2(\partial\Lambda)$, and the orthogonality conditions (A.5) hold, then there is a unique solution ϕ^0 to the problem (A.2)-(A.4) for any large positive constant K . Moreover there is a constant $C > 0$, independent of ε , such that the solution to the problem (A.2)-(A.4) satisfies a priori estimate*

$$\|\phi^0\|_{H^2(\Lambda)} \leq C \|G\|_{L^2(\partial\Lambda)}.$$

Proof. Since K is large, the proof of the existence and uniqueness of the solution to (A.2)-(A.4) and its estimate is standard. To show the L^2 -orthogonality (A.4), using the equations of $Z(x)$ and ϕ^0 and also the condition (A.5), for

$$\varphi(\theta, \eta) = \int_{\mathbb{R}} \phi^0(x, \theta, \eta) Z(x) dx,$$

one finds

$$\frac{1}{l_1^2(\theta)}\varphi_{\theta\theta} + \frac{1}{l_2^2(\theta)}\varphi_{\eta\eta} - (K - 1 - \lambda_0)\varphi = 0 \quad \text{in } \Gamma_\varepsilon, \quad \varphi_\eta(\theta, 0) = 0.$$

Choosing $K > \lambda_0 + 1$, we deduce that

$$\varphi(\theta, \eta) = \int_{\mathbb{R}} \phi^0(x, \theta, \eta) Z(x) dx = 0 \quad \text{in } \Gamma_\varepsilon.$$

Similarly we have

$$\int_{\mathbb{R}} \phi^0(x, \theta, \eta) w_x(x) dx = 0 \quad \text{in } \Gamma_\varepsilon.$$

□

A special case of Lemma A.1.1 is the following problem: finding function $\hat{\phi} \in H^2(\mathbf{\Lambda})$ such that

$$\begin{aligned} \hat{\phi}_{xx} + \frac{1}{l_1^2(\theta)} \hat{\phi}_{\theta\theta} + \frac{1}{l_2^2(\theta)} \hat{\phi}_{\eta\eta} - \tilde{K} \hat{\phi} + pw^{p-1} \hat{\phi} &= 0 \quad \text{in } \mathbf{\Lambda}, \\ \hat{\phi}_\eta &= G(x, \theta) \quad \text{on } \partial\mathbf{\Lambda}, \end{aligned} \tag{A.6}$$

where \tilde{K} is a large positive constant.

Lemma A.1.2. *Suppose the function $G(x, \theta)$ is even in the variable x , then there exists a large positive constant \tilde{K} such that the problem (A.6) has a unique solution $\hat{\phi}$, which is an even function in the variable x and satisfies*

$$\|\hat{\phi}\|_{H^2(\mathbf{\Lambda})} \leq C \|G\|_{L^2(\partial\mathbf{\Lambda})},$$

Moreover, if $G(x, \theta)$ is exponentially decaying in x , then

$$|\hat{\phi}(x, \theta, \eta)| < Ce^{-\alpha|x|}, \tag{A.7}$$

where $\alpha > 0$ and the constant C does not depend on ε .

Proof. For any \tilde{K} large enough, the proof of the existence and uniqueness of the solution to (A.6) and its estimate is standard. By uniqueness and evenness of $G(x, \theta)$, $\hat{\phi}$ is an even function in the variable x . By the exponentially decaying of $G(x, \theta)$, we also have (A.7). □

Next, we consider the following problem

$$\begin{aligned} \mathcal{L}_0(\tilde{\phi}) \equiv \tilde{\phi}_{xx} + \frac{1}{l_1^2(\theta)} \tilde{\phi}_{\theta\theta} + \frac{1}{l_2^2(\theta)} \tilde{\phi}_{\eta\eta} - \tilde{\phi} + pw^{p-1} \tilde{\phi} &= h \quad \text{in } \mathbf{\Lambda}, \\ \tilde{\phi}_\eta &= G(x, \theta) \quad \text{on } \partial\mathbf{\Lambda}, \\ \int_{\mathbb{R}} \tilde{\phi}(x, \theta, \eta) w_x(x) dx &= 0, \quad \int_{\mathbb{R}} \tilde{\phi}(x, \theta, \eta) Z(x) dx = 0 \quad \text{in } \Gamma_\varepsilon. \end{aligned} \tag{A.8}$$

Lemma A.1.3. *If $h \in L^2(\mathbf{\Lambda})$, $G \in L^2(\partial\mathbf{\Lambda})$ and the orthogonality conditions (A.5) hold, then for any solution $\tilde{\phi}$ to problem (A.8) we have*

$$\|\tilde{\phi}\|_{H^2(\mathbf{\Lambda})} \leq C [\|h\|_{L^2(\mathbf{\Lambda})} + \|G\|_{L^2(\partial\mathbf{\Lambda})}]$$

where the constant C does not depend on h , G and ε . Furthermore, if $|h| + |G| \leq Ce^{-\alpha|x|}$, then $|\tilde{\phi}| \leq Ce^{-c\alpha|x|}$ for some $C, c > 0$.

Proof. Let $\phi^0(x, \theta, \eta)$ be defined in Lemma A.1.1 and $\tilde{\phi} = \phi^0 + \phi$. Then we have

$$\begin{aligned} \phi_{xx} + \frac{1}{l_1^2(\theta)}\phi_{\theta\theta} + \frac{1}{l_2^2(\theta)}\phi_{\eta\eta} - \phi + pw^{p-1}\phi &= h + (1-K)\phi^0 \quad \text{in } \Lambda, \\ \phi_\eta &= 0 \quad \text{on } \partial\Lambda, \\ \int_{\mathbb{R}} \phi(x, \theta, \eta)w_x(x)dx &= 0, \quad \int_{\mathbb{R}} \phi(x, \theta, \eta)Z(x)dx = 0 \quad \text{in } \Gamma_\varepsilon. \end{aligned} \tag{A.9}$$

Let $\xi_k, k = 1, 2, \dots$, be the eigenfunctions (corresponding eigenvalues $\lambda_k, k = 1, 2, \dots$) of the following eigenvalue problem

$$-\frac{1}{l_1^2(\theta)}\phi_{\theta\theta} - \frac{1}{l_2^2(\theta)}\phi_{\rho\rho} = \lambda\xi \quad \text{in } \Gamma.$$

Let us consider Fourier series decompositions for h and ϕ of the form Let us consider Fourier series decompositions for $h + (1-K)\phi^0$ and ϕ :

$$\begin{aligned} \phi(x, \theta, \eta) &= \sum_{k=0}^{\infty} \phi_k(x)\xi_k(\varepsilon\theta, \varepsilon\eta), \\ h(x, \theta, \eta) + (1-K)\phi^0(x, \theta, \eta) &= \sum_{k=0}^{\infty} h_k(x)\xi_k(\varepsilon\theta, \varepsilon\eta). \end{aligned}$$

From the equation (A.9) we arrive at the following equation

$$-\varepsilon^2\lambda_k\phi_k + \phi_{k,xx} - \phi_k + pw^{p-1}\phi_k = h_k, \tag{A.10}$$

with the orthogonality condition

$$\int_{\mathbb{R}} \phi_k(x)w_x(x)dx = 0, \quad \int_{\mathbb{R}} \phi_k(x)Z(x)dx = 0. \tag{A.11}$$

Let us consider the bilinear form in $H^1(\mathbb{R})$

$$B(\psi, \psi) = \int_{\mathbb{R}} [|\psi_x|^2 + |\psi|^2 - pw^{p-1}|\psi|^2] dx.$$

Since (A.11) holds uniformly in k we conclude that

$$C[\|\phi_k\|_{L^2(\mathbb{R})}^2 + \|\phi_{k,x}\|_{L^2(\mathbb{R})}^2] \leq B(\phi_k, \phi_k),$$

for a constant $C > 0$ independent of k . Using this fact and equation (A.10) we arrive at

$$(1 + \lambda_k^2\varepsilon^4)\|\phi_k\|_{L^2(\mathbb{R})}^2 + \|\phi_{k,x}\|_{L^2(\mathbb{R})}^2 \leq C\|h_k\|_{L^2(\mathbb{R})}^2. \tag{A.12}$$

Moreover, we see from (A.10) that ϕ_k satisfies an equation of the form

$$\phi_{k,xx} - \phi_k = \tilde{h}_k \quad \text{on } \mathbb{R},$$

where $\|\tilde{h}_k\|_{L^2(\mathbb{R})} \leq C\|h_k\|_{L^2(\mathbb{R})}$. Hence it follows that

$$\|\phi_{k,xx}\|_{L^2(\mathbb{R})}^2 \leq C\|h_k\|_{L^2(\mathbb{R})}^2. \quad (\text{A.13})$$

Summing up estimates (A.12) and (A.13) in k , we conclude that

$$\|D^2\phi\|_{L^2(\Lambda)}^2 + \|D\phi\|_{L^2(\Lambda)}^2 + \|\phi\|_{L^2(\Lambda)}^2 \leq C\|h\|_{L^2(\Lambda)}^2.$$

The final estimate follows from the estimates of ϕ and ϕ^0 . \square

A corollary of Lemma A.1.3 is the following

Corollary A.1.1. *Let $G \in L^2(\partial\Lambda)$ satisfy the orthogonality conditions (A.5) and $h = 0$. Then problem (A.8) has a unique solution $\tilde{\phi}$ such that*

$$\|\tilde{\phi}\|_{H^2(\Lambda)} \leq C\|G\|_{L^2(\partial\Lambda)},$$

where the constant C does not depend on G and ε . Furthermore, if $|G| \leq Ce^{-\alpha|x|}$, then $|\tilde{\phi}| \leq Ce^{-c\alpha|x|}$ for some $C, c > 0$. \square

Appendix B. A Linear Model Problem II

Recall \mathfrak{S} represents the strip

$$\{ (x, z) : x \in \mathbb{R}, z \in \Gamma_\varepsilon \} \quad (\text{B.1})$$

in \mathbb{R}^3 . $\partial\mathfrak{S}$ is the component of the boundary of \mathfrak{S} , i.e.

$$\partial\mathfrak{S} = \{ (x, z) : x \in \mathbb{R}, z \in \partial\Gamma_\varepsilon \}.$$

We consider the following problem: given $h \in L^q(\mathfrak{S})$ and $G \in L^q(\partial\mathfrak{S})$, finding functions $\phi \in W^{2,q}(\mathfrak{S})$, $c, d \in L^q(\Gamma)$ and Λ_1, Λ_2 such that

$$\begin{aligned} \phi_{xx} + \Delta^{\Gamma_\varepsilon} \phi - \phi + pw^{p-1}\phi &= h + c(\varepsilon\eta)\chi(\varepsilon|x|)w_x + d(\varepsilon\eta)\chi(\varepsilon|x|)Z \quad \text{in } \mathfrak{S}, \\ \frac{\partial\phi}{\partial\tau_\varepsilon} &= G \quad \text{on } \partial\mathfrak{S}, \\ \int_{\mathbb{R}} \phi(x, z)w_x dx &= \Lambda_1, \quad \int_{\mathbb{R}} \phi(x, z)Z(x)dx = \Lambda_2 \quad \text{in } \Gamma_\varepsilon, \end{aligned} \quad (\text{B.2})$$

where τ_ε denotes the inward normal of $\partial\mathfrak{S}$, and $\chi(t)$ a smooth cut-off function such that $\chi(t) = 1$ for $|t| \leq 10\sigma$ and $\chi(t) = 0$ for $t \geq 20\sigma$, and $\sigma > 0$ is a small constant defined in Section 5.

Lemma B.2.4. *There exist functions $c(\varepsilon z)$, $d(\varepsilon z)$ with respect to h such that the problem (B.2) has a unique solution $\phi = T_1(h, G)$. Moreover,*

$$\begin{aligned} \|\phi\|_{q, \varrho} &\leq C(\|h\|_{L^q(\mathfrak{S})} + \|g\|_{L^q(\partial\mathfrak{S})}), \\ \|\Lambda_i\|_{L^q(\Gamma_\varepsilon)} &\leq C(\|h\|_{L^q(\mathfrak{S})} + \|g\|_{L^q(\partial\mathfrak{S})}), \quad \forall i = 1, 2, \end{aligned}$$

where the constant C does not depend on h , G and ε .

Proof. The proof will be carried out in three steps.

Step 1: Let us assume that in problem (B.2) the terms G , $c(\varepsilon z)$, $d(\varepsilon z)$ are identically zero. Arguing as in Lemma A.1.3, for any $h \in L^q(\mathfrak{S})$ and any solution $\phi \in W^{2,q}(\mathfrak{S})$ of problem (B.2) we have

$$\|\phi\|_{q,\varrho} \leq C \|h\|_{L^q(\mathfrak{S})} \quad (\text{B.3})$$

Step 2: We claim that the a priori estimate obtained in Step 1 is in reality valid for the full problem (B.2). We first choose suitable Λ_1 and Λ_2 such that

$$\nabla^{\Gamma_\varepsilon} \Lambda_1 = \int_{\mathbb{R}} g(x, z) w_x \, dx, \quad \nabla^{\Gamma_\varepsilon} \Lambda_2 = \int_{\mathbb{R}} g(x, z) Z(x) \, dx \quad \text{in } \Gamma_\varepsilon. \quad (\text{B.4})$$

Let ϕ^0 be the solution of

$$\phi_{xx}^0 + \Delta^{\Gamma_\varepsilon} \phi^0 - \phi^0 + p w^{p-1} \phi^0 = 0 \quad \text{in } \mathfrak{S}, \quad \frac{\partial \phi^0}{\partial \tau_\varepsilon} = G \quad \text{on } \partial \mathfrak{S},$$

Note that we have

$$\|\phi^0\|_{q,\varrho} \leq C \|G\|_{L^q(\partial \mathfrak{S})} \quad (\text{B.5})$$

By defining

$$\bar{\Lambda}_1 = \int_{\mathbb{R}} \phi^0(x, z) w_x \, dx, \quad \bar{\Lambda}_2 = \int_{\mathbb{R}} \phi^0(x, z) Z(x) \, dx \quad \text{in } \Gamma_\varepsilon,$$

we have that, to prove the general case it suffices to apply the argument with

$$\bar{\phi} = \phi - \phi^0 + \frac{(\bar{\Lambda}_1 - \Lambda_1) w_x}{\int_{\mathbb{R}} w_x^2 \, dx} + \frac{(\bar{\Lambda}_2 - \Lambda_2) Z(x)}{\int_{\mathbb{R}} Z^2(x) \, dx}.$$

The $\bar{\phi}$ satisfies a problem of a similar form with homogeneous Neumann boundary condition and orthogonality condition, as well as h replaced by a function \bar{h} with norm bounded by

$$\|\bar{h}\|_{L^q(\mathfrak{S})} \leq C (\|h\|_{L^q(\mathfrak{S})} + \|G\|_{L^q(\partial \mathfrak{S})}).$$

Step 3: We consider the problem

$$\begin{aligned} \bar{\phi}_{xx} + \Delta^{\Gamma_\varepsilon} \bar{\phi} - \bar{\phi} + p w^{p-1} \bar{\phi} &= \bar{h} + c \chi w_x + d \chi Z \quad \text{in } \mathfrak{S}, \\ \frac{\partial \bar{\phi}}{\partial \tau_\varepsilon} &= 0 \quad \text{on } \partial \mathfrak{S}, \\ \int_{\mathbb{R}} \bar{\phi}(x, z) w_x \, dx &= 0, \quad \int_{\mathbb{R}} \bar{\phi}(x, z) Z(x) \, dx = 0 \quad \text{in } \Gamma_\varepsilon. \end{aligned}$$

The existence of the solutions can be proved similarly as that in Lemma A.1.3. There also hold the priori estimate

$$\begin{aligned} \|\phi\|_{q,\varrho} &\leq C (\|h\|_{L^q(\mathfrak{S})} + \|g\|_{L^q(\partial \mathfrak{S})}), \\ \|\Lambda_i\|_{L^q(\Gamma_\varepsilon)} &\leq C (\|h\|_{L^q(\mathfrak{S})} + \|g\|_{L^q(\partial \mathfrak{S})}), \quad \forall i = 1, 2. \end{aligned}$$

□

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